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n branches are divided, there corresponds, in the expansion of H_1^k, a contribution to the coefficient h_α with multiplicity $k!/\beta_1!\beta_2! \cdots \beta_r!$. It follows then from Eq. (31)

$$h_\alpha = \frac{1}{(m-2)!} \sum_{k=1}^{n} \frac{(-1)^k}{k!} \times \sum_{(\beta_1\beta_2\cdots\beta_r)} \frac{k!}{\beta_1!\,\beta_2!\cdots\beta_r!} \bar{h}_1^{\beta_1} \cdots \bar{h}_r^{\beta_r}, \qquad (32)$$

in which $(\beta_1\beta_2 \cdots \beta_r)$ denotes the summation taken under the restriction $\beta_1 + \beta_2 + \cdots + \beta_r = k$. Now the coefficients $\bar{h}_1, \bar{h}_2, \cdots \bar{h}_r$ can be obtained by Lemma A. However it proves useful at this point to note that the factor $(n-1)!/\prod n_l!$ appearing in Eq. (24) is just the number of distinct ways to perform cyclic permutations on the group of n branches (among which $n_1, n_2, \cdots$ are identical). Using this interpretation of Lemma A for the expressions of $\bar{h}_1, \bar{h}_2, \cdots, \bar{h}_r$, we get

$$\bar{h}_1^{\beta_1}\bar{h}_2^{\beta_2} \cdots \bar{h}_r^{\beta_r} = (-1)^{n-k} \begin{Bmatrix}\text{product of the number of}\\ \text{distinct cyclic paramutations}\\ \text{of the branches in each group}\end{Bmatrix} \prod_l h_l^{n_l}. \qquad (33)$$

Substituting Eq. (33) into Eq. (32), we have

$$h_\alpha = \frac{1}{(m-2)!}(-1)^n M \prod_l h_l^{n_l}, \qquad (34)$$

where

$$M = \sum_{k=1}^{n} \sum_{(\beta_1\beta_2\cdots\beta_r)} \frac{1}{\beta_1!\,\beta_2!\cdots\beta_r!} \begin{Bmatrix}\text{product of the number of distinct cyclic}\\ \text{permutations of the branches in each}\\ \text{group}\end{Bmatrix}$$

$$= \sum_{k=1}^{n} \sum_{(\beta_1\beta_2\cdots\beta_r)} \begin{Bmatrix}\text{the number of distinct ways one can}\\ \text{permute the } n \text{ branches under the}\\ \text{particular grouping by performing}\\ \text{cyclic permutations within each}\\ \text{group while the groups are un-}\\ \text{numbered}\end{Bmatrix}$$

$$= \begin{Bmatrix}\text{the number of distinct ways one}\\ \text{can permute the } n \text{ branches by first}\\ \text{dividing into unnumbered groups and}\\ \text{then performing cyclic permutations}\\ \text{within each group}\end{Bmatrix}.$$

It is well-known that each permutation of a collection of objects can be analyzed into groups of cyclic permutations in an unique way. Therefore also

$$M = \begin{Bmatrix}\text{the number of distinct permutations of}\\ \text{the } n \text{ branches}\end{Bmatrix} = n!/\prod n_l!. \qquad (35)$$

The substitution of Eqs. (29) and (35) into Eq. (34) and the introduction of Eq. (30) now yields Eq. (25). Q.E.D.

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Strict Localization

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A complete characterization for a general quantum field theory is given of the strictly localized states introduced by J. Knight. It is shown that each such state can be generated from the vacuum by a partially isometric operator. Necessary and sufficient conditions are given for the superposition of such states to be also strictly localized. Finally, it is shown that there is a connection between the von Neumann type of the ring generated by the field operator in a finite region and the possibility of constructing strictly localized states.

I. INTRODUCTION

THE notion of strictly localized states has recently been introduced by Knight.[1] Let $\varphi(x)$ be a complete, local, scalar Hermitian field. Let Ω denote the vacuum state. Then a state Ψ is said to be strictly localized in a region G of Minkowski space if for any n,

$$\langle\Psi, \varphi(x_1) \cdots \varphi(x_n)\Psi\rangle = \langle\Omega, \varphi(x_1) \cdots \varphi(x_n)\Omega\rangle,$$

when all the $x_1 \cdots x_n$ are outside G.

Knight has shown, for the case of the free field, that such states cannot contain a finite number of particles. He has also shown that states of the form

$$e^{iA}\Omega,$$

where A is a smoothed polynomial in the field in a

[1] J. M. Knight, J. Math. Phys. 2, 459 (1961).

region G, are strictly localized in the union of the forward and backward light cones subtended by G.

In the following we will investigate strict localization in a general quantum field theory using the language of operator rings.[2,3] In Sec. II it will be shown that each strictly localized state can be generated from the vacuum by a certain partially isometric operator. In Sec. III, necessary and sufficient conditions are given for the superposition of such states to be also strictly localized. In Sec. IV, we show that there is a connection between the von Neumann type of the operator rings and the possibility of constructing strictly localized states.

II. STRICTLY LOCALIZED STATES

For any open region G in Minkowski space, we will denote by $R(G)$ the weakly closed symmetric ring of bounded operators generated by the projectors associated with the field $\varphi(x)$ in G.[4] The symbol G' will denote the spacelike complement of G. The commutant of the ring $R(G)$ will be denoted by $R'(G)$. The Hilbert space of physical states will be denoted by H.

In this notation, we have found it convenient to define strictly localized states as follows:

Definition 1. The state Ψ is said to be strictly localized *outside* the open region G, if for any $A \in R(G)$,

$$\langle \Psi, A\Psi \rangle = \langle \Omega, A\Omega \rangle. \tag{1}$$

A class of strictly localized states is given in the following theorem.

Theorem I. If W is any partially isometric operator in $R'(G)$ such that

$$W^\dagger W = 1, \tag{2}$$

then the state

$$\Psi = W\Omega \tag{3}$$

is strictly localized outside G.

Proof: Let $A \in R(G)$; consider

$$\langle \Psi, A\Psi \rangle = \langle \Omega, W^\dagger A W \Omega \rangle,$$

since $W \in R'(G)$,

$$\langle \Psi, A\Psi \rangle = \langle \Omega, A W^\dagger W \Omega \rangle,$$

by (2),

$$\langle \Psi, A\Psi \rangle = \langle \Omega, A\Omega \rangle.$$

If $W W^\dagger = 1$, then W is unitary, and we have the class of states considered by Knight. However, in general, $W W^\dagger = P \neq 1$, where P is some projector in $R'(G)$, as by (2),

$$P^2 = WW^\dagger WW^\dagger = P.$$

The operators W of Theorem I have a rather interesting property. If Φ_1, Φ_2 are any two states in H, and if A is any operator in $R(G)$, then

$$\langle W\Phi_1, AW\Phi_2 \rangle = \langle \Phi_1, A\Phi_2 \rangle. \tag{4}$$

The proof is the same as in Theorem I. This leads us to make the following definition.

Definition 2. A bounded operator W is said to be strictly localized outside the region G, if Eq. (4) holds for all $\Phi_1, \Phi_2 \in H$, and all $A \in R(G)$.

Theorem II. The bounded operator W is strictly localized outside the region G, if and only if

$$W^\dagger W = 1,$$

and

$$W \in R'(G).$$

Proof: The sufficiency is immediate. We will prove the necessity. Equation (4), since it holds for any $\Phi_1, \Phi_2, \in H$, implies that

$$W^\dagger A W = A, \tag{5}$$

for all $A \in R(G)$. In particular, for $A = 1$,

$$W^\dagger W = 1. \tag{6}$$

However, for any $A \in R(G)$,

$$W^\dagger A W = W^\dagger[A, W] + W^\dagger W A = W^\dagger[A, W] + A,$$

by (6). Thus

$$W^\dagger[A, W] = 0. \tag{7}$$

Similarly, we find that

$$[W^\dagger, A]W = 0. \tag{8}$$

Now, since $A \in R(G)$, then $A^\dagger$ and $A^\dagger A \in R(G)$, and by (5),

$$W^\dagger A^\dagger A W = A^\dagger A = [W^\dagger, A^\dagger][A, W] + [W^\dagger, A^\dagger]WA + A^\dagger W^\dagger[W, A] + A^\dagger W^\dagger W A.$$

Using Eqs. (6), (7), and (8), we get

$$([A, W])^\dagger[A, W] = 0.$$

Let $\Phi \in H$. This implies that

$$|[A, W]\Phi|^2 = 0, \quad \text{and} \quad [A, W]\Phi = 0,$$

which implies, since Φ is arbitrary,

$$[A, W] = 0.$$

Thus, $W \in R'(G)$.

Theorems I and II together imply that if W is an

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[2] R. Haag and B. Schroer, J. Math. Phys. 3, 248 (1962).
[3] M. A. Naimark, *Normed Rings*, translated from the 1st Russian edition by L. F. Boron (P. Noordhoff, Ltd., Groningen, The Netherlands, 1959).
[4] H. Reeh and S. Schlieder (to be published).

operator strictly localized outside a region G, then the state $W\Omega$ is strictly localized outside G. The next theorem will show that all strictly localized states are of this form.

Theorem III. To every state Ψ, strictly localized outside an open region G, there corresponds an operator W, strictly localized outside G, such that

$$W\Omega = \Psi.$$

Proof: To prove this theorem we will need the following Lemma.

Lemma. If G is an open region in Minkowski space, then

$$\text{closure } (R(G)\Omega) = H.$$

Proof: Let $\mathfrak{A}(G)$ denote the operator algebra generated by finite polynomials in the field $\varphi(x)$, smoothed by testing functions with support in G. Reeh and Schlieder[5] have shown that

$$\text{closure } (\mathfrak{A}(G)\Omega) = H. \quad (9)$$

They have also shown[4] that

$$\mathfrak{A}''(G) = R(G),$$

and that this with (9) implies the above Lemma.

Now let Φ be any element of H. By the Lemma, there exists a sequence $\{A_n, A_n \in R(G), n=1, 2 \cdots\}$, such that

$$A_n\Omega \to \Phi, \quad (10)$$

i.e.,

$$|A_n\Omega - \Phi| \to 0.$$

Consider the expression

$$\Psi(\Phi) = \lim_{n\to\infty} A_n\Psi. \quad (11)$$

This limit exists, for

$$|A_n\Psi - A_m\Psi| = |(A_n - A_m)\Psi|,$$

Since Ψ is localized outside G,

$$\Psi(\Phi) = |(A_n - A_m)\Omega| \to 0,$$

by Eq. (9). The space H is complete. Thus $\Psi(\Phi)$ exists and is unique. The limit (11) is also independent of the particular sequence $\{A_n\}$. For if $\{A_n'\}$ is another such sequence,

$$|A_n\Psi - A_n'\Psi| = |(A_n - A_n')\Psi| = |(A_n - A_n')\Omega| \leq |(A_n\Omega - \Phi)| + |(A_n'\Omega - \Phi)| \to 0.$$

[5] H. Reeh and S. Schlieder, Nuovo Cimento **22**, 1051 (1961).

Thus $\Psi(\Phi)$ depends only on the states Ψ and Φ. We claim that it is linear in Φ. For if $\{A_n^1\}$ and $\{A_n^2\}$ are two sequences in $R(G)$ such that

$$A_n^i\Omega \to \Phi^i, \qquad i = 1, 2,$$

then clearly, for any complex numbers α, β,

$$(\alpha A_n^1 + \beta A_n^2)\Omega \to \alpha\Phi^1 + \beta\Phi^2,$$

and

$$\Psi(\alpha\Phi^1 + \beta\Phi^2) = \lim_{n\to\infty} ((\alpha A_n^1 + \beta A_n^2)\Psi) = \alpha \lim_{n\to\infty} A_n^1\Psi + \beta \lim_{n\to\infty} A_n^2\Psi = \alpha\Psi(\Phi^1) + \beta\Psi(\Phi^2).$$

The correspondence

$$\Phi \to \Psi(\Phi)$$

thus defines a unique linear operator W, depending only on Ψ, such that

$$\Psi(\Phi) = W\Phi.$$

This operator is bounded, for

$$|W\Phi| = \lim_{n\to\infty} |A_n\Psi| = \lim_{n\to\infty} |A_n\Omega| = |\Phi|.$$

We claim that

$$W\Omega = \Psi.$$

For, the sequence $\{A_n = 1\}$ is such that

$$A_n\Omega \to \Omega,$$

but now

$$W\Omega = \lim_{n\to\infty} A_n\Psi = \Psi.$$

The operator W is strictly localized outside G. For let $\Phi_1, \Phi_2 \in H$, $A \in R(G)$. Then there exist sequences $\{A_n^i\} \subset R(G)$, $i = 1, 2$, such that

$$A_n^i\Omega \to \Phi^i, \qquad i = 1, 2.$$

Consider

$$\langle W\Phi^1, AW\Phi^2\rangle = \lim_{n,m\to\infty} \langle A_n^1\Psi, AA_m^2\Psi\rangle = \lim_{n,m\to\infty} \langle\Psi, A_n^{1\dagger}AA_m^2\Psi\rangle,$$

since Ψ is strictly localized outside G,

$$\langle W\Phi^1, AW\Phi^2\rangle = \lim_{n,m\to\infty} \langle\Omega, A_n^{1\dagger}AA_m^2\Omega\rangle = \langle\Phi^1, A\Phi^2\rangle.$$

Corollary 1. Let Ψ be strictly localized outside G.

The operator W constructed above is unique in the sense that it is the only operator in $R'(G)$ such that $W\Omega = \Psi$. It is not, however, the only operator which can create Ψ from the vacuum.

Proof: Suppose $V \in R'(G)$ and $V\Omega = \Psi$. Then for any $A \in R(G)$,

$$A(W - V)\Omega = 0 = (W - V)A\Omega.$$

Since states of the form $A\Omega$ are dense in H, this implies that

$$W = V.$$

Now the operator WP_Ω, where P_Ω is the projector onto the vacuum, certainly generates Ψ from the vacuum. We cannot, however, have

$$WP_\Omega = W,$$

for then, premultiplying both sides of this equation by $W^\dagger$, we would obtain

$$P_\Omega = 1,$$

a contradiction.

Given a state Ψ strictly localized outside G, the problem now arises as to whether there exists some region F such that the associated operator W is in $R(F)$. The solution rests on the validity of the duality theorem, which has been partially proved by Haag and Schroer.[2] This theorem states that for any open region G,

$$R'(G) = R(G'),$$

from which it immediately follows that $F = G'$.

III. SUPERPOSITION OF STRICTLY LOCALIZED STATES

With the aid of the associated localized operators, it is now possible to give the conditions under which it is possible to superimpose localized states.

Theorem IV. Let Ψ_1, Ψ_2 be two states strictly localized outside an open region G. Let W_1, W_2 be the corresponding strictly localized operators. Then the states

$$\Psi(\alpha, \beta) = N[\alpha\Psi_1 + \beta\Psi_2], \tag{12}$$

where α and β are any complex numbers and

$$N = |\alpha\Psi_1 + \beta\Psi_2|^{-1},$$

will be strictly localized outside G if and only if

$$W_2^\dagger W_1 = (\Omega, W_2^\dagger W_1\Omega) = r, \quad \text{say.} \tag{13}$$

Proof: We will prove the sufficiency first. Suppose (13) holds. Then the operator

$$W = N(\alpha W_1 + \beta W_2)$$

is in $R'(G)$, and is such that

$$\begin{aligned} W^\dagger W = (|\alpha|^2 + |\beta|^2 + \alpha\beta^* r + \alpha^*\beta r^*)^{-1} \\ \times [|\alpha|^2 W_1^\dagger W_1 + |\beta|^2 W_2^\dagger W_2 \\ + \alpha\beta^* W_2^\dagger W_1 + \alpha^*\beta W_1^\dagger W_2] \\ = 1. \end{aligned}$$

By Theorem I, the state

$$\Psi(\alpha, \beta) = W\Omega$$

is strictly localized outside G.

We will prove now the necessity. Suppose that $\Psi(\alpha, \beta)$ is strictly localized outside G. Then for any $A \in R(G)$,

$$\langle\Psi(\alpha, \beta), A\Psi(\alpha, \beta)\rangle = \langle\Omega, A\Omega\rangle.$$

Using the expression (12), and the strict locality of the states Ψ_1 and Ψ_2, we find that

$$\begin{aligned} \alpha^*\beta\langle\Omega, W_1^\dagger A W_2\Omega\rangle + \alpha\beta^*\langle\Omega, W_2^\dagger A W_1\Omega\rangle \\ = (\alpha^*\beta r^* + \beta^*\alpha r)\langle\Omega, A\Omega\rangle. \end{aligned}$$

Since this must hold for all α, β, we conclude that

$$\langle\Omega, W_2^\dagger A W_1\Omega\rangle = r\langle\Omega, A\Omega\rangle.$$

Now suppose $A = A_1^\dagger A_2$, for arbitrary $A_i \in R(G)$, $i = 1, 2$. Since $W_i \in R'(G)$,

$$\langle A_1\Omega, W_2^\dagger W_1 A_2\Omega\rangle = r\langle A_1\Omega, A_2\Omega\rangle.$$

But by the Reeh–Schlieder Lemma, states of the form $A\Omega$, $A \in R(G)$ are dense in H. This implies, by continuity,

$$\langle\Phi_1, W_1^\dagger W_1\Phi_2\rangle = r\langle\Phi_1, \Phi_2\rangle$$

for all Φ_1, $\Phi_2 \in H$. Thus,

$$W_2^\dagger W_1 = r.$$

Corollary. If the states $\Psi(\alpha, \beta)$ are strictly localized outside G, then we may express one of the operators W_i, W_2 say, in the form

$$W_2 = r^* W_1 + (1 - |r|^2)^{\frac{1}{2}} U. \tag{14}$$

where the operator U is strictly localized outside G, and takes H into a subspace orthogonal to the subspace $W_1 H$.

Proof: The number r is in magnitude less than 1, as

$$|r| = |\langle\Omega, W_2^\dagger W_1\Omega\rangle| \leq |\Omega|^2\, |W_2^\dagger|\, |W_1| = 1.$$

Thus the root in (14) is real. The operator U is clearly in $R'(G)$. By Eq. (13),

$$U^\dagger W_1 = 0. \tag{15}$$

Since $W_2^\dagger W_2 = 1$, we get

$$U^\dagger U = 1, \text{ i.e., } \quad U \text{ is partially isometric.}$$

Thus U is strictly localized outside G. Consider the projectors

$$P_1 = W_1 W_1^\dagger, \qquad P = UU^\dagger.$$

From (15) it follows that

$$PP_1 = 0.$$

Thus W_1 and U take H into mutually orthogonal subspaces.

IV. VON NEUMANN TYPES

There has been some interest expressed recently[2] in determining the von Neumann factor types of the rings $R(G)$. The following theorem has a physical interpretation which makes factor type III seem most reasonable.

Theorem V. Suppose that

(a) The Hilbert space H is separable.

(b) For any open region G, the ring $R(G)$ is a factor, i.e., $R(G) \cap R'(G) = (\alpha 1)$.

Then to each projector $P \in R(G)$ there corresponds a partially isometric operator $W \in R(G)$, such that $W^\dagger W = 1$, $P = WW^\dagger$, if and only if the factor $R(G)$ is of von Neumann type III.

Proof: Assumption (a) has generally been assumed to be true.[2] A partial proof of assumption (b) based on primitive causality has been given by Haag and Schroer.[2]

Let us suppose that the factor $R(G)$ is of type III. Then the relative dimension function D, defined on projectors in $R(G)$, takes on only the values 0, ∞.[6] It is 0 only for the null projector. This implies that all nonnull projectors in $R(G)$ project onto infinite subspaces. The separability of H then implies that all such projectors are equivalent,[7] in particular, equivalent to 1, i.e., that $P \in R(G)$, $P = 0$ implies the existence of a $W \in R(G)$ such that

$$P = WW^\dagger, \qquad W^\dagger W = 1.$$

Thus we have proven the sufficiency of the above condition.

Consider now the necessity. If to every nonnull projector $P \in R(G)$ there corresponds an operator $W \in R(G)$ such that

$$W^\dagger W = 1, \qquad WW^\dagger = P,$$

then each such projector is equivalent to the unit operator. This implies that for $P \neq 0$,[8]

$$D(P) = D(1).$$

There are only two von Neumann types compatible with this condition—type I_1, or type III. The type I_1 must however be ruled out, as it would imply that $R(G)$ consisted of only multiples of the identity.[9]

This theorem may be interpreted as follows: To each proposition q concerning a measurement made by an apparatus located in the region G, there is associated a projector $P \in R(G)$,[10] such that for any state $\Phi \in H$, $P\Phi = \Phi$ if q is true in Φ, but $P\Phi = 0$ if it is false. A partially isometric operator $W \in R(G)$ such that $W^\dagger W = 1$, $WW^\dagger = P$, we interpret as representing an apparatus in G, which alters any state such that the proposition corresponding to P is true, i.e.,

$$PW\Phi = WW^\dagger W\Phi = W\Phi.$$

Moreover, by Theorem II, the state $W\Phi$ will not differ from Φ as far as measurements made in G' are concerned.

With this interpretation, Theorem V is equivalent to the statement that the ring $R(G)$ will be of von Neumann type III if and only if for every proposition q in G, there is an apparatus in G which can alter any state such that q is true, without affecting any measurement made in the region G'.

Note added in proof. The author is indebted to Professor H. Araki for the following comments.

(1) The notion of strict localization is a special case of the notion of "equivalence". Two states are said to be "equivalent in a region G", if their expectation values are the same for all operators in $R(G)$. A state equivalent to the vacuum in G is then strictly localized outside G.

(2) Theorem III can be proven more directly by defining an operator V by

$$VA\Omega = A\Psi$$

for all $A \in R(G)$. The closure of this operator can be shown to exist, and to have the required properties.

(3) Theorem V can also be interpreted as saying that $R(G)$ is of type III if and only if for any projector $P \in R(G)$, and any state $\Psi \in H$, there is an eigenstate of P that is equivalent to Ψ in G'.

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[6] Reference 3, p. 469, Theorem 2.

[7] Reference 3, p. 457, Proposition VI.

[8] Reference 3, p. 465, Eq. (3**).

[9] Reference 3, p. 483, Theorem 2.

[10] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer-Verlag, Berlin, 1932) [English edition: Princeton University Press, Princeton, New Jersey, 1955].