Algebraic quantum field theory: objectives, methods, and results

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Abstract

Algebraic quantum field theory is a general mathematical framework for relativistic quantum physics, based on the theory of operator algebras. It comprises all observable and operational aspects of a theory. In its framework the entire state space of a theory is covered, starting from the vacuum over arbitrary configurations of particles to thermal equilibrium and non-equilibrium states. It provides a solid foundation for structural analysis, the physical interpretation of the theory and the development of new constructive schemes. This survey is commissioned by the *Encyclopedia of Mathematical Physics*, edited by M. Bojowald and R.J. Szabo. It is to be published by the Elsevier publishing house.

1 Origin and achievements

Algebraic quantum field theory (AQFT) emerged from the framework of quantum field theory [44,E1], which relies on the principle of locality (the quantum version of Maxwell's Nahwirkungsprinzip). The primary goal of quantum field theory is the description of relativistic particles and their interactions. But there appeared several questions.

(a) Non-interacting field theories were known to describe particles that propagate freely. But how can one extract from a theory its particle aspects in the presence of interaction?

(b) Quantum physics exhibits non-local phenomena, such as entanglement and the violation of Bell inequalities. How is this compatible with the locality principle, in particular with the maximum signal velocity of light?

(c) It was realized that there are theories with different field content and different Lagrangians which describe the same physics. Can one identify physically indistinguishable theories by some meaningful equivalence relation?

(d) Are there properties of quantum field theories that are generic, *i.e.* independent of specific models?

Algebraic Quantum Field Theory answered these and related questions of physical interest in a general framework, based on first principles. It arose from the problem of computing a scattering matrix for the multitude of elementary particles and their compounds that were observed in high energy experiments. The idea to assign to each of these interacting particles a separate quantum field, as in non-interacting theories, seemed to be odd. This problem was solved by Haag [33, II.4]. He observed that it is sufficient for the computation of scattering matrices to exhibit for each one-particle state suitable operators, built out of a few fundamental fields, which have non-vanishing matrix elements between the vacuum and the one-particle state. There exist many operators with this property in general, but the resulting scattering matrices do not depend on their specific choice.

Based on this insight, Haag proposed to reformulate quantum field theory as follows [33, III]: instead of dealing with the technically subtle point-localized quantum fields, one takes as basic input the algebras of genuine operators generated by fields in bounded subregions of spacetime. Thereby one focuses on combinations of the fields describing observables. This restriction allows one to postulate properties of direct physical significance for the resulting algebras. As a matter of fact, it turned out to be irrelevant that the algebras are generated by quantum fields.

The fundamental paradigm is that the linking of algebras with spacetime regions uniquely characterizes a theory. The mathematical framework of AQFT is put on a rigorous basis by the Haag-Kastler axioms [35], which rely on Einstein causality and the Poincaré symmetry of Minkowski space. Later it was extended to generally covariant theories on globally hyperbolic spacetimes. Here we restrict our attention to theories on four-dimensional Minkowski space.

A crucial feature of AQFT, describing systems with an infinite number of degrees of freedom, is the existence of disjoint representations of the algebras by Hilbert space operators. They describe macroscopically distinguishable systems differing, for example, by temperature, global charges, or their behavior at infinity. This fact required an overview of the representations, in particular the identification of representations describing elementary states, such as the vacuum and single particle states.

Vacuum representations are characterized by the existence of a Poincaré invariant ground state; these representations may also describe single particle states, such as the photon. But charged single particle states do not belong to vacuum representations of the observables. The corresponding charged representations were characterized by Doplicher, Haag, and Roberts [23] by the property that they cannot be distinguished from the vacuum representation by observables which are localized in the spacelike complement of given bounded regions. This excludes states with electrical charge because of Gauss' law, but includes states with nonzero baryon or lepton number. We say, these states carry a localizable charge. With this input, Doplicher, Haag and Roberts were able to completely unravel the structure of the corresponding representations. They clarified the origins of Bose and Fermi statistics, the existence of global gauge groups that describe the charges of particles, and they constructed charged Bose and Fermi fields that interpolate between the vacuum and the charged states [23,24,26]. The dimension of physical spacetime played an important role in this context. In two spacetime dimensions other forms of statistics and symmetries can occur.

The analysis of gauge theories, in particular of the confinement problem, made it clear that it is not sufficient to consider only particles with localizable charges. Another approach, taken by Buchholz and Fredenhagen [16], therefore started directly with the discussion of representations, where the bottom of the energy-momentum spectrum contains some isolated mass hyperboloid, characterizing the states of a single particle. There then exists an associated vacuum representation which coincides with the particle representation on observables localized in the complement of cone-shaped regions extending to spacelike infinity. Hence in massive theories all particles can be localized in such cones. These weaker localizability properties still allowed it to arrive at results similar to those obtained by Doplicher, Haag and Roberts for localizable charges. Other types of statistics and symmetries can appear for cone-localizable states already in three spacetime dimensions.

In theories describing long range forces, such as quantum electrodynamics, the charged particle structure is less well understood. The reason is that the particle states always contain infinite clouds of low energy massless particles. For them a meaningful classification is not available. However, in physical spacetime one can make use of the fact that massless particles propagate with the speed of light (Huygens principle). Thus the probability of finding an unlimited number of them in a given forward lightcone, is equal to zero. One therefore restricts the states to the observables in lightcones with fixed apex and concentrates on the properties of the resulting representations. In case of charged states carrying a simple charge, like the electric one, one arrives in this manner at similar results as in massive theories. In particular, the charge structures are described by abelian gauge groups [20].

There are other states of great physical interest which describe thermal equilibria. Since in infinite space Hamiltonians of physical interest have continuous spectrum, the Gibbs-von Neumann description of equilibrium states by density matrices in the vacuum representation can not be used, the thermal representations are disjoint from the vacuum representation. Instead, one characterizes the equilibrium states by specific analyticity properties of their correlation functions, the so-called KMS condition, named after Kubo, Martin, and Schwinger and invented by Haag, Hugenholtz, and Winnink [34, E2]. In order to identify those theories which admit equilibrium states for all positive temperatures, one needs criteria which restrict the size of state space [21, 36]. These criteria impose constraints on the nature of correlations between observations in spacelike separated regions. As a matter of fact, the correlations can be completely suppressed in suitable states [12, 13]. This feature allows it to construct equilibrium states in finite spacetime regions which coincide in their spacelike complements with the vacuum. Thinking of high energy physics, it resembles the formation of a quark-gluon plasma. Enlarging the regions occupied by these localized equilibrium states, they have thermodynamic limits which satisfy the KMS condition [18].

There is a surprising relationship between the KMS condition in algebraic quantum field theory and a cornerstone in the theory of operator algebras, *viz.* modular theory, invented by Tomita and Takesaki [46, E3]. In the mathematical theory one studies suitable algebras of operators on a Hilbert space, which do not contain elements (apart from 0) that annihilate a given state. It turns out that these algebras have an intrinsic time evolution for which the correlation functions of the given state satisfy the KMS condition. Thermal states in AQFT fit into this framework. But the mathematical results also imply, for example, that the vacuum state is a KMS states with regard to the intrinsic dynamics of the algebra of observables localized in a wedge region, bounded by two lightlike planes. The intrinsic time evolution is in this case the one-parameter group of Lorentz boosts, which leaves the wedge invariant. In physical terms, uniformly accelerated observers register in the vacuum state some non-zero temperature (Unruh effect) [43].

The connection of AQFT with modular theory led to a variety of fruitful applications that could not be achieved within the conventional formalism of quantum fields. It helped to establish duality relations (Haag-duality) between observables in spacelike separated regions, which is crucial for the analysis of the properties of superselection sectors [3]. It was also used in arguments establishing the universal structure of local algebras [29]. A remarkable property of these algebras is the absence of finite dimensional projections. This must be kept in mind when discussing measurements and operations. Arguments that claim their apparent acausal behavior often ignore this fact and are therefore invalid. As a matter of fact, modular theory has been a key ingredient in the discussion of entanglement. Last, but not least, it has become an important tool in the rigorous constructions of models which are not accessible by other methods, such as theories with factorizing scattering matrices [40, E4].

These results confirmed the conviction that the physical content of a theory is encoded in the net structure of the underlying local algebras of observables. The question of how to characterize within the general formalism a specific theory (*e.g.* by the reconstruction of a corresponding Lagrangian function from the algebras) remained open, however. Only recently it has turned out that a complementary view on this issue is more fruitful [17]. Regarding fields and their composites, such as Lagrangian functions, as classical objects, corresponding quantum operations can rigorously be defined. It provides for given Lagrangian function, involving local interactions, a corresponding net of local algebras satisfying the basic postulates of AQFT. The scheme works for theories involving Bose as well as Fermi fields and leads, for example, to an algebraic understanding of renormalization and related issues, such as the appearance of anomalies [9, 10]. The prospects for a general constructive scheme within AQFT that applies to all theories of physical interest, including gauge theories, are promising.

In the following the underlying notions, specific methods, and key results are outlined in greater detail. References to articles in this encyclopedia have an E in front of their number. Other references are given by their numbers only.

2 The algebraic framework

As in any quantum theory, the observables and resulting basic operations are described in AQFT by elements A of some non-commutative associative algebra $\mathfrak{A}$ over the complex numbers $\mathbb{C}$ together with an antilinear involution $A \mapsto A^*$. The algebra $\mathfrak{A}$ may be thought of as being concretely given in some defining representation by operators acting on a Hilbert space, where A^* is the adjoint of A. But, as an abstract algebra, it has a multitude of other representations by Hilbert space operators. One usually refers to $\mathfrak{A}$ as algebra of observables.

Observables correspond to selfadjoint elements $A = A^* \in \mathfrak{A}$, and basic operations are described by unitaries $U \in \mathfrak{A}$. Their adjoint action $\mathrm{ad}\, U(A) := UAU^*$ on any observable A does not change its spectrum and corresponding multiplicities. Assuming that the spectrum of the observables A is bounded, which can always be accomplished by a suitable (non-linear) choice of scale, they have a bounded norm $\|A\|$, inherited from the defining Hilbert space representation. This norm satisfies for any $A \in \mathfrak{A}$ the condition $\|A^*A\| = \|A\|^2$, the so-called C*-property. Completing $\mathfrak{A}$ with respect to this norm, one obtains a C*-algebra [33, Sect. III.2.1]. It corresponds to the norm closure of a subalgebra of the algebra of all bounded operators in the defining representation.

It is a distinctive property of AQFT that it comprises information as to where and when measurements are made, *i.e.* about their localization in Minkowski space $\mathscr{M}$. (The metric on $\mathscr{M}$, used here, is positive on timelike vectors, the velocity of light is $c = 1$.) Thus, given any bounded spacetime region $\mathscr{O} \subset \mathscr{M}$, there is a corresponding subalgebra $\mathfrak{A}(\mathscr{O}) \subset \mathfrak{A}$ containing all operators which correspond to measurements or operations within the ranges of $\mathscr{O}$. As these sets of operators become bigger if the localization region increases, one has the inclusions

$$\mathfrak{A}(\mathscr{O}_1) \subset \mathfrak{A}(\mathscr{O}_2) \quad \text{if} \quad \mathscr{O}_1 \subset \mathscr{O}_2 \, . \tag{2.1}$$

Due to this property of isotony, the mapping of spacetime regions to algebras, $\mathscr{O} \mapsto \mathfrak{A}(\mathscr{O})$, constitutes a net based on Minkowski space $\mathscr{M}$ from which the algebra $\mathfrak{A}$ of all observables can be recovered in the limit $\mathscr{O} \nearrow \mathscr{M}$. The principle of Einstein causality implies that observations in spacelike separated regions must be commensurable. Whence, because of the finite propagation speed of physical effects, they cannot disturb each other in any way. This basic fact is encoded in the condition of locality,

$$[\mathfrak{A}(\mathscr{O}_1), \mathfrak{A}(\mathscr{O}_2)] = \{0\} \quad \text{if} \quad \mathscr{O}_1 \perp \mathscr{O}_2 \,. \tag{2.2}$$

It says that the commutator of any pair of observables which are localized in spacelike separated regions, indicated by $\mathscr{O}_1 \perp \mathscr{O}_2$, has to vanish.

Relativity enters by assuming that the symmetry group of Minkowski space $\mathscr{M}$, the proper orthochronous Poincaré group $\mathscr{P}_+^\uparrow = \mathscr{L}_+^\uparrow \ltimes \mathbb{R}^4$, acts by automorphisms $\alpha(\lambda)$ on $\mathfrak{A}$, $\lambda \in \mathscr{P}_+^\uparrow$. Recalling that the elements of $\mathscr{P}_+^\uparrow$ relate inertial observers to each other, the action of the automorphisms on the local algebras satisfies

$$\alpha(\lambda)(\mathfrak{A}(\mathscr{O})) = \mathfrak{A}(\lambda\mathscr{O}) \,, \quad \lambda \in \mathscr{P}_+^\uparrow \,, \tag{2.3}$$

in an obvious notation. Thus the localization regions of the algebras change under the automorphisms according to the geometric action of the corresponding Poincarè transformations on Minkowski space. These physically meaningful conditions define a mathematical framework for the observables and operations in any physically acceptable relativistic quantum theory on Minkowski space [33, III]. It provides a basis for their general analysis, their physical interpretation, and it paves the way for the construction of models.

3 Algebraic constructions

At present, the existence of interacting quantum fields in physical spacetime has been accomplished only by perturbation theory, *i.e.* field operators are defined in terms of formal power series whose convergence properties are not under control. These methods have been refined and transferred to the algebraic setting, leading to perturbative AQFT [8, 42, E5]. Quite recently, the insights gained in these investigations led to a new constructive scheme [17]. It yields for given classical Lagrangian function a concrete algebra which complies with all postulates of AQFT. This construction is outlined in the following for scalar selfinteracting quantum fields.

The ingredients in this approach are arbitrary classical, real, and smooth scalar fields $x \mapsto \phi(x)$ on Minkowski space $\mathscr{M}$, which may be unbounded at infinity.

Corresponding classical local observables are described by functionals of the form

$$\phi \mapsto F[\phi] := \sum_{n=0}^{N} \int dx\, f_n(x)\phi(x)^n\,, \tag{3.1}$$

where $x \mapsto f_n(x)$ are real test functions with compact support in $\mathscr{M}$. The support of a functional F in Minkowski space is defined as the union of the supports of the underlying test functions f_n, $n = 1, \dots, N$. The functional for $n = 0$ has empty support. It can be assigned to any given spacetime region.

Lagrangian functions (densities) on $\mathscr{M}$, describing local self-interactions of the classical field, are of the form

$$x \mapsto L(x)[\phi] := (1/2)\,(\partial_\mu \phi(x)\,\partial^\mu \phi(x) - m^2 \phi(x)^2) - \sum_{j=1}^{J} g_j\, \phi(x)^j\,, \tag{3.2}$$

where ∂_μ denotes the partial derivative with regard to the μ-coordinate of x, m is some mass value, and g_j are real coupling constants. Integrating these Lagrangian functions over all of $\mathscr{M}$, whenever meaningful for a given field ϕ, yields a value of its action.

However, the integral defining the action will in general not converge. But local variations of the action can always be defined. To this end one introduces shifts of the functionals F, putting $F^{\phi_0}[\phi] := F[\phi + \phi_0]$, where $x \mapsto \phi_0(x)$ are arbitrary real and smooth scalar fields with compact support. Applying these shifts to the Lagrangian functions, one finds that $x \mapsto (L(x)[\phi + \phi_0] - L(x)[\phi])$ can be integrated over all of $\mathscr{M}$ for any field ϕ. Thus the variations of the corresponding actions given by

$$\delta L(\phi_0)[\phi] := \int dx\,(L(x)[\phi + \phi_0] - L(x)[\phi]) \tag{3.3}$$

are well defined for all fields ϕ. As a matter of fact, one finds by partial integration that $\delta L(\phi_0)$ is a functional of the form (3.1) for fixed ϕ_0.

Based on this classical input, one defines for given Lagrangian function L a corresponding dynamical group $\mathscr{G}_L$ that is associated with AQFT. It is the group generated by elements $S_L(F)$, satisfying two basic relations, where F are arbitrary functionals as defined in equation (3.1).

(1) Identifying the localization of $S_L(F)$ in Minkowski space with the support of the corresponding functional F, a first equality describes causal relations between

these elements. Whenever the support of a functional F_1 is later than that of a functional F_2, *i.e.* there is some Cauchy surface in $\mathscr{M}$ such that F_1 lies above and F_2 beneath it, one has

$$S_L(F_1)S_L(F_2) = S_L(F_1 + F_2)\,. \tag{3.4}$$

Note that the product on the left hand side is not commutative unless the supports of the functionals are spacelike separated; only then can one find Cauchy surfaces separating the supports in either temporal order. Hence relation (3.4) is an expression of relativistic causality which is more stringent than the condition of locality. This relation also implies that the constant functionals $\phi \mapsto c[\phi] := c$, $c \in \mathbb{R}$, determine elements $S_L(c)$ of the center of $\mathscr{G}_L$. Moreover, they satisfy the equation $S_L(c_1)S_L(c_2) = S_L(c_1 + c_2)$.

(2) The second relation, involving the dynamical input determined by the Lagrangian function L, is given by

$$S_L(F) = S_L(F^{\phi_0} + \delta L(\phi_0)) \tag{3.5}$$

for all functionals F and shift fields ϕ_0. This equality is an integrated version of the Schwinger-Dyson equation for quantum fields, where the given Lagrangian function L enters [17,E5].

Heuristically, the elements $S_L(F)$ can be interpreted as basic operations which describe the impact of perturbations induced by F on the underlying systems. They may be thought of as time ordered exponentials of $(i/\hbar)F$ acting on a quantum field. But no explicit quantization procedures are used in their construction. As a matter of fact, the dynamical group $\mathscr{G}_L$ is non-commutative from the outset due to the arrow of time that enters in the defining relation (3.4).

We want to interpret the operations $S_L(F)$ as unitary operators acting on Hilbert spaces. To this end we put $S_L(c) := e^{ic}\,1$ for the constant functionals $c \in \mathbb{R}$, which is compatible with relations (3.4) and (3.5). It amounts to choosing atomic units, where Planck's constant is $\hbar = 1$. A simple example of such a Hilbert space with an action of the operations can be constructed as follows. The space is generated by the complex linear span of vectors $|S\rangle$, $S \in \mathscr{G}_L$, where $|S_L(c)S\rangle := e^{ic}|S\rangle$. The scalar product for these generating vectors is given by

$$\langle S|S'\rangle := \begin{cases} e^{ic} & \text{if } S' = S_L(c)S = e^{ic}S \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

The action of the group $\mathscr{G}_L$ on this space is defined by $S|S'\rangle := |SS'\rangle$. Thus the elements $S \in \mathscr{G}_L$ act as invertible isometric operators and hence are unitary.

All linear combinations of these unitaries, which by the distributive law for products form an algebra $\mathfrak{A}_L$, are faithfully represented on this space, *i.e.* do not vanish unless they are identically 0. To see this, consider the sum $\sum_k c_k S_k$, where group elements differing only by some phase factor are combined in a single term with an appropriate c-number factor. Applying this operator to the vector $|1\rangle$, the resulting vector has the norm-square $\sum_k |c_k|^2$ and hence vanishes only if all coefficients c_k are equal to 0, as claimed. So the operator norm on the Hilbert space determines a C*-norm on $\mathfrak{A}_L$. In a similar manner, any other faithful Hilbert space representation of $\mathfrak{A}_L$ determines a norm on it. Since the C*-norm of any unitary operator is equal to 1, the supremum of all C*-norms on $\mathfrak{A}_L$ is well defined and one can equip the algebra with this maximal C*-norm and complete it in this topology. The result is a C*-algebra, which is denoted by the same symbol.

The local subalgebras $\mathfrak{A}_L(\mathscr{O}) \subset \mathfrak{A}_L$ are generated by elements $S_L(F)$, where F has support in $\mathscr{O} \subset \mathscr{M}$. So the net $\mathscr{O} \mapsto \mathfrak{A}_L(\mathscr{O})$ satisfies the condition of locality. There exist also automorphisms on the net which induce Poincaré transformations $\mathscr{P}_+^\uparrow$. This follows from the fact that the underlying classical Lagrangian functions transform as scalar fields under their action. Hence $\mathfrak{A}_L$ satisfies all Haag-Kastler axioms.

This approach not only leads to a rigorous construction of theories fitting into the general framework of AQFT for a large set of Lagrangians, but it also provides a basis for computations. We briefly illustrate this fact in the simple case of a non-interacting Lagrangian L_0, where all coupling constants in equation (3.2) are put equal to 0. One considers for arbitrary real test functions f on $\mathscr{M}$ the functionals $F_W(f)$ of the form

$$\phi \mapsto F_W(f)[\phi] := (1/2) \int dxdy\, f(x)\Delta_D(x-y)f(y) + \int dz\, f(z)\phi(z)\,, \tag{3.7}$$

where $\Delta_D = (1/2)\,(\Delta_R + \Delta_A)$ is the mean of the retarded and advanced solutions of the Klein-Gordon equation with mass m. The first term on the right hand side of (3.7) defines some constant functional, the second one is linear in the underlying field.

Putting $W(f) := S_{L_0}(F_W(f))$ and making use of relations (3.4) and (3.5) yields after some computations the equalities [17]

$$W(f_1)W(f_2) = e^{-(i/2)\int dxdy\, f_1(x)\Delta(x-y)f_2(y)}\, W(f_1+f_2)\,,$$
$$W((\square+m^2)f_3) = 1\,. \tag{3.8}$$

Here f_1, f_2, f_3 are real test functions, $\square$ is the d'Alembertian, and $\Delta := (\Delta_R - \Delta_A)$ (Pauli-Jordan function). Thus the operators $W(f)$ are unitary exponentials of a real, scalar, local quantum field that satisfies the Klein-Gordon equation and is integrated with test functions f (Weyl operators). These well known operators are elements of the algebra $\mathfrak{A}_{L_0}$ and appear in the present approach without imposing any commutation relations from the outset. The exponent of the phase factor reveals the use of atomic units, where $\hbar = 1$.

We conclude this outline by noting that a more refined version of the causality relation (3.4), involving higher products, has further interesting consequences [17]. For example, given any Lagrangian L of the form (3.2), the corresponding net of local algebras $\mathscr{O} \mapsto \mathfrak{A}_L(\mathscr{O})$ consists of subalgebras of the global algebra $\mathfrak{A}_{L_0}$, determined by the non-interacting Lagrangian L_0; but these subalgebras differ from the local algebras in the non-interacting theory. In other words, different theories merely differ by the resulting nets in a fixed global algebra. In this way one arrives at an algebraic version of the interaction picture. Vacuum representations for different Lagrangians are, however, inequivalent in agreement with Haag's Theorem.

The algebraic constructions, explained here in simple cases, have been extended to theories involving an arbitrary finite number of interacting bosonic and fermionic fields. Moreover, several questions of physical interest have been settled. Among them are the time evolution of given initial data (time slice axiom), the occurrence of symmetries (Noether's theorem), and the impact of changes of renormalization (renormalization group) [9, 10, E5].

This new constructive approach has contributed to the consolidation of AQFT, showing that C*-algebras satisfying the Haag Kastler axioms exist in presence of interaction. An important future step will be the analysis of their state spaces. Given a dynamical algebra, one may be able to show that it has a vacuum state, in analogy to the existence proofs of constructive quantum field theory. Or one may be able to show that for a given dynamics the corresponding algebra does not have any such state, cf. [1, 31, E6].

4 States and representations

The algebras $\mathfrak{A}$, complying with the basic assumptions of AQFT, are designed to explore the properties of all ensembles which can appear in a theory. Dealing with quantum theory, one makes statistical predictions about measuring results. These are encoded in expectation functionals $\omega : \mathfrak{A} \to \mathbb{C}$, *i.e.* linear maps from the elements of the algebra to complex numbers. Each ensemble in which a system can be prepared determines some functional ω. Given $A \in \mathfrak{A}$, the entity $\omega(A)$ is interpreted as expectation value (mean of measuring results) in the corresponding ensemble. Since the variance of an observable cannot be negative and A^*A is an observable with non-negative spectrum, one demands $\omega(A^*A) \geq 0$ for $A \in \mathfrak{A}$. One also requires the normalization $\omega(1) = 1$. Any such positive and normalized functional ω on $\mathfrak{A}$ is called a state.

It is an important fact that every state ω determines a representation of the algebra $\mathfrak{A}$, the GNS representation [33, Sect. III.2], which acts by operators on some Hilbert space. It is given by a triple $(\pi, \mathscr{H}, \Omega)$ consisting of a Hilbert space $\mathscr{H}$, a unit vector $\Omega \in \mathscr{H}$ representing the given state, and a structure preserving map (homomorphism) $\pi : \mathfrak{A} \to \mathscr{B}(\mathscr{H})$ of the algebra $\mathfrak{A}$ into the algebra $\mathscr{B}(\mathscr{H})$ of bounded operators on $\mathscr{H}$. The link between the two settings is provided by the formula

$$\omega(A) = \langle \Omega, \pi(A)\Omega \rangle, \quad A \in \mathfrak{A}. \tag{4.1}$$

Thus expectation functionals of elements of $\mathfrak{A}$ are represented by matrix elements of corresponding operators on $\mathscr{H}$. Given a state ω, the corresponding GNS-representation is unique, up to unitary equivalence. Dealing with infinite systems, different states in a theory lead in general to quite different (inequivalent) Hilbert space representations. Hence the usage of the concept of state is more flexible than starting with some particular Hilbert space representation of the algebra.

Given a state ω, one obtains other states by two physically meaningful operations. The first one corresponds to perturbations, which are caused by measurements or by operations such as local changes of the dynamics. They are described by operators $V \in \mathfrak{A}$ which are scaled such that $\omega(V^*V) = 1$. The resulting states are given by $A \mapsto \omega_V(A) := \omega(V^*AV)$, $A \in \mathfrak{A}$. The second operation is the formation of mixtures of these perturbed states, which are described by convex combinations. The norm closure of the resulting convex set of states is called the folium of ω. All states in this folium can be described by density matrices in the GNS–representation induced by ω.

Two states are said to be disjoint if their respective folia have an empty intersection. In that case there exist classical observables distinguishing the folia. They are obtained by sequences of observables whose commutators with any other observable tend to 0 and whose expectation values converge to different numbers in the two folia. Prominent examples are global charges, temperature, and order parameters distinguishing different phases. The situation is particularly simple in the case of pure states, *i.e.* states which cannot be decomposed into convex combinations of other states. Pure states describe ensembles with maximal information and induce irreducible representations. The folia of any two pure states are either disjoint or they coincide. One therefore speaks of sectors of the entire state space which are formed by these folia.

Let us finally mention some important technical point in this context. If a sequence $\{A_n \in \mathfrak{A}\}_{n \in \mathbb{N}}$ converges in norm to some operator A, this entails the uniform convergence of its expectation values,

$$\lim_n \sup_\omega |\omega(A_n - A)| = 0\,, \tag{4.2}$$

where the supremum is taken over all states. In a given folium one can consider a weaker form of convergence. One demands that for any state in the folium, the expectation values converge to those of some bounded operator A, which may not be contained in $\mathfrak{A}$, however. Completing the local algebras in this weak topology yields von Neumann algebras (also called W*-algebras) in the representation fixed by the folium. These completions contain for example the projection operators appearing in the spectral decomposition of local observables, which enter in the probabilistic interpretation of the underlying states. Different folia induce in general different weak topologies, so it is not meaningful to work with this topology from the outset.

5 Elementary states

On any C*-algebra $\mathfrak{A}$ there exists an abundance of states and corresponding representations. Many of them are of limited physical interest since they describe over-idealizations, such as infinite accumulations of matter. It is therefore of importance to identify those states which are essential for the physical interpretation of the theory.

Conceptually, the simplest states are vacuum states [33, III.4]. A vacuum state ω_0 (there may be several such states or none) is by definition a ground state which looks alike for all inertial observers. It therefore is invariant under Poincaré transformations, $\omega_0\,\alpha(\lambda) = \omega_0$, $\lambda \in \mathscr{P}_+^\uparrow$, where the product of the state and the Poincaré automorphism indicates their composition. This implies that there exists in the GNS-representation $(\pi_0, \mathscr{H}_0, \Omega_0)$, induced by ω_0, a unitary representation U_0 of the Poinaré group which is given by, $A \in \mathfrak{A}$,

$$U_0(\lambda)\,\pi_0(A)\,\Omega_0 := \pi_0(\alpha(\lambda)(A))\,\Omega_0\,, \quad \lambda \in \mathscr{P}_+^\uparrow\,. \tag{5.1}$$

The correlation functions $\lambda \mapsto \omega_0(A^*\alpha(\lambda)(A))$ are supposed to be continuous, $A \in \mathfrak{A}$, so the unitary representation $\lambda \mapsto U_0(\lambda)$ is weakly continuous. Hence the subgroup of unitaries representing the spacetime translations $x \in \mathbb{R}^4$ can be presented by Stone's theorem in the form $U_0(x) = e^{ixP}$ with generators P, which are interpreted as energy-momentum operators. Their joint spectrum can be determined by Fourier analysis of the correlation functions. Since a vacuum state is a ground state for all inertial observers, the spectrum must be contained in the forward lightcone $V_+ = \{p \in \mathbb{R}^4 : p_0 \geq 0,\, p^2 \geq 0\}$, where p_0 denotes the energy component and p^2 the Minkowski square of p. Finally, vacuum states can always be uniquely decomposed into a convex combination of disjoint pure vacuum states. We restrict our attention in the following to pure vacuum states and assume that the corresponding GNS-representations are faithful, *i.e.* they are regarded as defining representation of $\mathfrak{A}$.

Vacuum states have many interesting properties. Among them are clustering properties of vacuum correlation functions [2], the absence of local operators annihilating the vacuum (Reeh-Schlieder property [41]), the entanglement of spacelike separated operations [45, E7], and the energetic effects of the spontaneous breakdown of internal symmetries (algebraic Goldstone theorem [15]). Because of lack of space, most of these results cannot be discussed here in detail.

The folium of the vacuum state contains only states which are neutral in the sense that they can be obtained by applying observables to the vacuum vector, a prominent example being the photon. Charged particles lead, by definition, to disjoint representations of the underlying algebra.

The determination of the charged particle content of a theory is based on two physical ideas. The first one, going back to Haag and Kastler [35], is to consider sequences of states in the folium of the vacuum that consist, heuristically, of a charge in a fixed region and a compensating charge in another region which is moved to spacelike infinity. The compensating charge at infinity does then no longer contribute to the expectation values, so the limit states carry only the original charge. The second idea has been advocated by Borchers [4] and is based on the condition that elementary states of interest should be pure and admit a representation of the group of spacetime translations with generators having joint spectrum in V_+. This spectrum is automatically invariant under Lorentz transformations as consequence of locality [5]. In theories where only massive particles appear, one can identify charged single particle states by the condition that the spectrum does not contain the discrete point $p = 0$, corresponding to the vacuum, but an isolated mass shell $p^2 = m^2$ for some mass $m > 0$. As will be discussed, this input can be used for the construction and analysis of composite states containing several particles.

This approach fails, however, in theories with long range forces, such as quantum electrodynamics. There particle states carrying an electric charge can neither be localized in bounded regions of Minkowski space because of the Coulomb field which they carry along. Nor is their mass shell separated from the rest of the spectrum due to clouds of low energy photons which inevitably accompany them as a consequence of Gauss's law and locality. There is progress in the identification of such particles [19, E8, E9], but this topic deserves further studies.

6 Sectors, statistics, and charged fields

Having identified the elementary charged states in a theory, a number of subsequent questions arise. First, given two charged states, does there exist a state containing both charges (addition of charges)? Second, does there exist for each charged state a state carrying the opposite charge (charge conjugation)? Third, can one assign to charged states a particular statistics (Bose-Fermi alternative)? And forth, do there always exist charged fields which create the charged states from the vacuum and transform as tensors under the action of some global gauge group? These questions require an answer, since one has initially only the observable algebra $\mathfrak{A}$ of a theory at ones disposal and charge-carrying fields are not given from the outset.

For localizable charges, all of these questions have found an affirmative answer in AQFT. Starting from the characterization of elementary states as local excitations of the vacuum, Doplicher, Haag and Roberts have established these facts in extensive investigations [23, 24, 26]. These were later supplemented by Buchholz and Fredenhagen for theories containing exclusively massive particles [16]. The latter results cover theories, where certain particle states fail to be localized excitations of the vacuum since they carry gauge or topological charges. The central ideas and methods underlying these results are outlined in the following.

In order to characterize elementary states ω on a given algebra $\mathfrak{A}$ which are local excitations of a vacuum state ω_0, one compares the respective expectation values of observables that are localized in distant regions. Given a causally closed region $\mathscr{O}$, which has the form of a double-cone (intersection of a forward and a backward lightcone), let $\mathscr{O}^c$ be its spacelike complement. The corresponding subalgebra $\mathfrak{A}(\mathscr{O}^c) \subset \mathfrak{A}$ is defined as the algebra generated by all local observables having their support in double-cones contained in $\mathscr{O}^c$. The state ω is then said to describe a local excitation of ω_0 if for every $\varepsilon > 0$ there exists some sufficiently large double cone $\mathscr{O}$ such that

$$\|\omega - \omega_0\|_{\mathscr{O}^c} := \sup_{A \in \mathfrak{A}(\mathscr{O}^c),\ \|A\|=1} |\omega(A) - \omega_0(A)| < \varepsilon\,. \tag{6.1}$$

In simple terms, the two states cannot be distinguished by measurements at large spacelike distances.

It is a consequence of relation (6.1) that the states ω and ω_0, restricted to $\mathfrak{A}(\mathscr{O}^c)$, induce equivalent representations of this subalgebra even though they are disjoint on $\mathfrak{A}$. Moreover, the GNS representation of $\mathfrak{A}$ induced by ω is faithful. Using methods of the theory of operator algebras, it follows that it can be represented on the vacuum Hilbert space $\mathscr{H}_0$ in a manner such that it coincides with the vacuum representation on $\mathfrak{A}(\mathscr{O}^c)$. This representation can be continued to the weak closures of the local algebras in the vacuum representation. For the sake of simple notation, we denote in this section the weak closures of $\pi_0(\mathscr{A}(\mathscr{O}))$ by $\mathfrak{R}(\mathscr{O}) := \pi_0(\mathscr{A}(\mathscr{O}))^-$ and the inductive norm limit of the resulting net $\mathscr{O} \mapsto \mathfrak{R}(\mathscr{O})$ by $\mathfrak{R}$. We also assume that the algebras $\mathfrak{R}(\mathscr{O})$ are maximal in the sense that they contain every operator which commutes with all operators $A^c \in \pi_0(\mathscr{A}(\mathscr{O}^c))$ ("Haag duality"). The representation of the extended algebra $\mathfrak{R} \supset \pi_0(\mathfrak{A})$, induced by ω, is denoted by $(\rho, \mathscr{H}_0, \Omega_0)$, that is

$$\omega(R) = \langle \Omega_0, \rho(R)\,\Omega_0 \rangle\,, \quad R \in \mathfrak{R}\,. \tag{6.2}$$

Recalling that the GNS-representations are unique up to unitary equivalence, the vector $\Omega_0 \in \mathscr{H}_0$, which represents ω_0 in the vacuum representation, now represents the charged state ω in the representation ρ. The localization properties of ω

and the feature of Haag duality imply that the represented algebra $\rho(\mathfrak{R})$ is not just any subalgebra of bounded operators on $\mathscr{H}_0$: it is contained in the original domain $\mathfrak{R}$ of ρ. Moreover, the representation ρ coincides with the vacuum representation on the algebra $\mathfrak{R}(\mathscr{O}^c)$,

$$\rho(R) = R \quad \text{if} \quad R \in \mathfrak{R}(\mathscr{O}^c)\,, \tag{6.3}$$

revealing the fact that the charge can be localized in the bounded region $\mathscr{O}$.

In view of these properties, the first question raised above has an immediate answer: given representations ρ_1 and ρ_2 that are induced by elementary charged states, a representation containing both charges is obtained by their composition $\rho_1\rho_2$. This composition is well defined since the ranges of the representations are contained in their domains $\mathfrak{R}$. The answer to the second question, concerning the existence of opposite charges, is immediate for the family of representations ρ induced by states carrying so-called simple charges [23]. The corresponding representations ρ are distinguished by the property that they map the algebra $\mathfrak{R}$ onto itself, so they have an inverse ρ^{-1}. An example of a simple charge is univalence, which distinguishes Bosons from Fermions. The representation ρ^{-1} is localized in the same region as ρ and compensates the charge of ρ by composition. Only these simple charges are discussed in the following; an outline of the general case of non-simple charges, where $\rho(\mathfrak{R})$ is a proper subalgebra of $\mathfrak{R}$, would require more space.

Coming to the third question concerning statistics, one makes use of the fact that the considered representations are translation covariant, so translations of the charges do not change their values. To see this, let us recall that the translations in the vacuum representation act by unitaries U_0 on the vacuum Hilbert space $\mathscr{H}_0$; their adjoint action on $\mathfrak{R}$, denoted by α_0, leaves this algebra invariant. Similarly, the translations in representations ρ induced by elementary states act by unitaries U_ρ on $\mathscr{H}_0$, hence their adjoint action is defined on $\mathfrak{R}$ as well. Given a representation ρ with charge contained in $\mathscr{O}$, as explained above, the translated representation with charge contained in $\mathscr{O}+x$ is given by

$${}^x\rho := \alpha_0(x)\,\rho\,\alpha_0(-x) = \mathrm{ad}\,(U_0(x)U_\rho(-x))\,\rho\,, \quad x \in \mathbb{R}^4\,. \tag{6.4}$$

Thus the representations ρ and ${}^x\rho$ are unitarily equivalent, whence they carry the same charge. The unitaries (called charge transporters) which connect the representations are elements of the local algebra $\mathfrak{R}(\mathscr{O}_{0,x})$, where $\mathscr{O}_{0,x}$ is any double cone containing $\mathscr{O}$ and $\mathscr{O}+x$. So they are elements of the algebra $\mathfrak{R}$ for all localizable charges. It is noteworthy that the adjoint action of the unitaries $U_\rho(x)U_0(-x)$

on $\mathfrak{R}$ allows one to recover in the vacuum sector the charged representation ρ in the limit of large spacelike $x \in \mathbb{R}^4$. This procedure corresponds to the heuristic idea of creating charged states from bi-localized neutral states.

The statistics of a charged representation ρ is determined by analyzing the products (compositions) ${}^x\rho{}^y\rho$ for translations $x, y \in \mathbb{R}^4$. One first notices that the resulting representations are all equivalent to the representation $\rho^2 := \rho\rho$ since the charge-transport operators are elements of $\mathfrak{R}$. In the case of simple charges, considered here, ρ^2 maps $\mathfrak{R}$ onto itself again. One then considers the products for translations x, y such that the charge of ${}^x\rho$ in $\mathscr{O} + x$ is spacelike separated from the charge of ${}^y\rho$ in $\mathscr{O} + y$. It is an important consequence of the locality property of the observables that ${}^x\rho{}^y\rho = {}^y\rho{}^x\rho$, *i.e.* the creation of charges in spacelike separated regions does not depend on their order [23]. Putting $\Gamma(x) := U_0(x)U_\rho(-x)$, one has

$$ {}^x\rho{}^y\rho = \mathrm{ad}\,(\Gamma(x))\,\rho\,\mathrm{ad}\,(\Gamma(y))\rho = \mathrm{ad}\,(\Gamma(x)\rho(\Gamma(y))\,\rho^2\,, \quad x, y \in \mathbb{R}^4\,, \tag{6.5} $$

since the composition of maps is associative. Now $\rho^2(\mathfrak{R}) = \mathfrak{R}$ is irreducibly represented in the representation induced by the pure vacuum state. Hence relation (6.5) implies that the unitary charge-transport operators $\Gamma(x)\rho(\Gamma(y))$, respectively $\Gamma(y)\rho(\Gamma(x))$, shifting the two charges in ρ^2 into spacelike separated regions $\mathscr{O} + x$, respectively $\mathscr{O} + y$, can only differ by some phase factor $\varepsilon_\rho(x, y)$,

$$ \Gamma(x)\rho(\Gamma(y)) = \varepsilon_\rho(x, y)\Gamma(y)\rho(\Gamma(x))\,, \quad \mathscr{O} + x \perp \mathscr{O} + y\,. \tag{6.6} $$

In a final deformation argument one uses the fact that in four spacetime dimensions one can exchange continuously any two spacelike separated double cones whilst keeping them spacelike apart. This fact entails that the phase factors do not depend on the admissible translations, *i.e.* one has $\varepsilon_\rho := \varepsilon_\rho(x, y)$ for all such x, y. Moreover, one obtains for their square $\varepsilon_\rho^2 = 1$. Thus there exists for each sector of a simple charge, described by a representation ρ, a corresponding unique number $\varepsilon_\rho \in \{\pm 1\}$, called statistics parameter.

The statistics parameters enter in the commutation relations of field operators that render the representations ρ. We briefly indicate their construction for representations describing self-conjugate charges; there the representation ρ^2 is equivalent to the vacuum representation, *i.e.* $\rho^2 = \mathrm{ad}\,V$ for some unitary operator $V \in \mathfrak{R}(\mathscr{O})$. In this case the field operators are defined on the direct sum of the neutral and the charged representation space, described by the vectors (Φ, Ψ), where $\Phi, \Psi \in \mathscr{H}_0$. The observables and translations are represented there by

$$\boldsymbol{R} := \begin{pmatrix} R & 0 \\ 0 & \rho(R) \end{pmatrix}, \; R \in \mathfrak{R}, \qquad \boldsymbol{U}(x) := \begin{pmatrix} U_0(x) & 0 \\ 0 & U_\rho(x) \end{pmatrix}, \; x \in \mathbb{R}^4. \tag{6.7}$$

One then defines unitary field operators $\boldsymbol{F}$ on this space, putting

$$\boldsymbol{F} := \begin{pmatrix} 0 & 1 \\ V & 0 \end{pmatrix}, \qquad \boldsymbol{F}^* = \begin{pmatrix} 0 & V^* \\ 1 & 0 \end{pmatrix}, \tag{6.8}$$

where $V \in \mathfrak{R}(\mathscr{O})$ is the unitary operator given above. This yields, in an obvious notation,

$$\boldsymbol{F}\boldsymbol{R}\boldsymbol{F}^* = \begin{pmatrix} \rho(R) & 0 \\ 0 & VRV^* \end{pmatrix} = \begin{pmatrix} \rho(R) & 0 \\ 0 & \rho^2(R) \end{pmatrix} = \rho(\boldsymbol{R}), \quad R \in \mathfrak{R}. \tag{6.9}$$

Hence the field implements the action of ρ and commutes with all observables in $\mathfrak{R}(\mathscr{O}^c)$. In this sense, it is localized in $\mathscr{O}$. Putting $\boldsymbol{F}(x) := \boldsymbol{U}(x)\boldsymbol{F}\boldsymbol{U}(x)^*$ for $x \in \mathbb{R}^4$, it follows that

$$\boldsymbol{\Gamma}(x) := \boldsymbol{F}(x)\boldsymbol{F}^* = \begin{pmatrix} U_0(x)U_\rho(x)^* & 0 \\ 0 & U_\rho(x)VU_0(x)V^* \end{pmatrix} = \begin{pmatrix} \Gamma(x) & 0 \\ 0 & \rho(\Gamma(x)) \end{pmatrix}. \tag{6.10}$$

Thus $\boldsymbol{\Gamma}(x)$ comprises the charge transfer operators of the representations ρ, respectively $\rho^2 = \mathrm{ad}V$. The properties of these transfer operators, established in relation (6.6), lead to the equality for all x, y such that $\mathscr{O} + x \perp \mathscr{O} + y$

$$\boldsymbol{F}(x)\boldsymbol{F}(y) = \boldsymbol{\Gamma}(x)\rho(\boldsymbol{\Gamma}(y))\boldsymbol{F}^2 = \varepsilon_\rho \boldsymbol{\Gamma}(y)\rho(\boldsymbol{\Gamma}(x))\boldsymbol{F}^2 = \varepsilon_\rho \, \boldsymbol{F}(y)\boldsymbol{F}(x). \tag{6.11}$$

So, depending on the sign of ε_ρ, the field operators localized at spacelike distances satisfy Bose, respectively Fermi, commutation relations. Which sign appears is encoded in the net structure of the underlying observable algebra.

Let us finally remark that there exists a unitary representation of the group $\mathbb{Z}_2$ on the underlying representation space which is given by

$$\boldsymbol{Z} = \boldsymbol{Z}^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.12}$$

Its adjoint action leaves all observables $\boldsymbol{R}$ invariant and changes the sign of the fields $\boldsymbol{F}$, so it is an example of a unitary representation of a global gauge group. To summarize, proceeding in the present example from two disjoint elementary states on the algebra of observables, one obtains an extension of the observable algebra by local field operators, satisfying either Bose or Fermi commutation relations at spacelike distances. They generate from the vacuum state, together with the observables, a Hilbert space containing all charged and neutral states. Using them, one can then compute collision states and scattering matrices [24, E8], such as in standard quantum field theory. However, in contrast to the standard setting, the present approach takes into account all particles occurring in a theory, including those for which no fields were initially specified. This is essential for proofs of asymptotic completeness [E8].

After this survey of methods underlying sector analysis, let us summarize the wealth of results which have been obtained on this topic so far. First, these investigations were performed for all kinds of localized elementary states [24]. The resulting representations ρ then no longer need to have an inverse; they may be morphisms which map the observable algebra into itself, not onto. The products of these morphisms can generically be decomposed into finite direct sums of morphisms which induce irreducible representations of the observables on the underlying vacuum Hilbert space. Every such irreducible morphism determines a statistics parameter that indicates its Bose, respectively Fermi (para)statistics. Moreover for any such morphism ρ there exists a conjugate morphism $\overline{\rho}$ whose product with ρ contains the vacuum representation ι.

The construction of field operators in this general framework turned out to be difficult and required advanced methods from category theory. It was finally accomplished by Doplicher and Roberts [26]. They proved that for the collection of localized morphisms in a theory there exists a unique extension of the observable algebra by compactly localized field operators, resulting in a field algebra. That algebra generates from the vacuum a representation space of the observables, describing all finite configurations of charged and neutral elementary systems. On this space, there acts a faithful unitary representation of a compact group under whose action the fields transform as tensors, while the observables remain fixed. Thus this group operates as a global gauge group. The fields satisfy Bose, respectively Fermi commutation relations at spacelike distances. Finally, there exists on the representation space a continuous unitary representation of the covering group of the Poincaré group with positive energy. Its adjoint action on the fields transforms their localization regions covariantly, in accordance with their statistics. So also in this general case the structures have been established which were taken for granted in standard quantum field theory.

There remained, however, the question whether this analysis covers all situations of physical interest in theories with short range forces, describing exclusively massive particles at large scales. This question was raised and answered by Buchholz and Fredenhagen [16]. Without any *a priori* assumption about localization properties, they proceeded from an elementary charged state on the algebra of observables. If the bottom of the energy-momentum spectrum in the resulting representation consists of some isolated mass shell, signaling a single particle, there exists also an accompanying vacuum state. The states in the sector of the particle are excitations of the vacuum which can be localized in spacelike cones $\mathscr{S} \subset \mathscr{M}$ that extend to spacelike infinity. More precisely, relation (6.1) holds for the norm distance between the two states if the region $\mathscr{O}^c$ is replaced by $\mathscr{S}^c$, the spacelike complement of any spacelike cone $\mathscr{S}$ containing some sufficiently big neighborhood $\mathscr{O}$ of the origin.

Given such a particle state and spacelike cone $\mathscr{S}$, one obtains as in case of localizable charges a representation of the algebra of observables which acts on the vacuum Hilbert space and is identical to the vacuum representation on the algebra $\mathfrak{A}(\mathscr{S}^c)$. But in general it does not map the full algebra of observables into itself, so defining the composition of these representations requires more work. Remarkably, all basic results obtained for localizable charges can also be established in this case. Among them are the Bose, respectively Fermi, (para)statistics of sectors, the existence of conjugate sectors, of a field algebra generated by charged field operators satisfying Bose, respectively Fermi commutation relations, and of a global compact gauge group acting on the fields and leaving the observables invariant. Moreover, the possibility of infinite statistics, which was originally left open in case of localizable charges, was proven not to occur in massive particle theories [28]. These results cover massive gauge theories, where non-confined charged particles appear at asymptotic time, or particles carrying a fluctuating topological charge.

As already mentioned, a complete understanding of the sector structure and statistics has not yet been accomplished for theories describing long range forces between local observables. The reasons are the infinite clouds of massless particles which are created by these interactions. They lead to an abundance of disjoint infrared sectors which cannot be discriminated in experiments. In other words, the theoretical concept of sectors is too subtle in these cases. In order to isolate the physically relevant structures, Buchholz and Roberts proposed to form equivalence classes of sectors, making use of the fact that infrared sectors cannot be distinguished by local observables in any given lightcone [20]. This feature fits with the fact that one cannot make up measurements in the past, and it explains why infrared sectors do not play a role in experiments.

Based on these insights, Buchholz and Roberts performed an investigation of the sector structure of the algebras of observables localized in any given lightcone. Restricting attention to sectors which are excitations of a vacuum state in spacelike (hyper)cones, they succeeded in identifying the sectors carrying simple charges. They then established the fact that these sectors have the same properties with regard to statistics, charge conjugation, existence of a global gauge group, and of charged fields as the sectors in massive theories in Minkowski space. These results complement in some sense an investigation of localizable charges in arbitrary globally hyperbolic spacetimes [32, E10]. It would be desirable to extend it to all elementary cone-localized systems in lightcones in order to determine their possible structures.

We conclude this section by noting that investigations of the sector structure have also been performed in low spacetime dimensions. There other types of statistics can appear, described by representations of the braid group. The sector structure can then in general no longer be described by the representation theory of a compact group, it is determined by other group-like structures. We refer the interested reader to [E11].

7 Thermal states and modular theory

Elementary systems play a fundamental role in the interpretation of the microscopic properties of a theory. One then aims to extract from it its macroscopic features, such as the properties of thermal equilibrium states. Conversely, the requirement that a theory ought to have a decent macroscopic behavior leads to constraints on its microscopic structure. We outline in this section some basic results in this respect and indicate a few consequences of physical interest. A more extensive discussion of equilibrium states can be found in [E2].

In the standard approach to equilibrium states one proceeds from finitely extended systems, *e.g.* confined in a box, and considers Gibbs-von Neumann ensembles whose density matrix is described by exponential functions of the negative Hamiltonian, multiplied with the inverse of the temperature $T > 0$. This works whenever the level density of the system is such that the trace of this operator is finite. That feature disappears, however, if one proceeds to the thermodynamic limit. Dealing in AQFT from the outset with infinitely extended systems, there arises the question whether one can recover from it this level density, which encodes information about the number of degrees of freedom in a theory.

The idea for such a procedure goes back to Haag and Swieca [36]; it was later refined by Buchholz and Wichmann [21]. One considers excitations of the vacuum state ω_0 on the observable algebra $\mathfrak{A}$ which are localized in given bounded spacetime regions $\mathscr{O}$ and suppresses their high energy contributions. Thereby, one obtains in the resulting GNS-representation linear maps $\theta_{T,\mathscr{O}} : \mathfrak{A}(\mathscr{O}) \to \mathscr{H}_0$,

$$\theta_{T,\mathscr{O}}(A) := e^{-(1/T)H} \pi_0(A)\Omega_0 , \quad A \in \mathfrak{A}(\mathscr{O}) , \tag{7.1}$$

where H is the generator of the time translations in a chosen Lorentz frame. Restricting these maps to operators in the unit ball of $\mathfrak{A}(\mathscr{O})$, one determines their ranges in the vacuum Hilbert space $\mathscr{H}_0$. Heuristically, one considers the smallest Hilbert-box into which such a range fits, *i.e.* an infinite dimensional cuboid in $\mathscr{H}_0$, centered at 0, which contains it. Of primary interest are theories, where the sums of the side lengths of these cuboids are finite; these sums replace the partition functions of finite systems. One can then define corresponding norms $\|\theta_{T,\mathscr{O}}\|_1$ of the maps. If these norms are finite, the maps are said to be nuclear.

Theories, where these norms exhibit a physically meaningful behavior in the limit of large temperatures and regions, have been shown to admit thermal equilibrium states on the algebra of observables for all temperatures [18]. An important intermediate step in the proof is the demonstration that these theories have the so-called split property [13,25]: given any pair of bounded regions $\mathscr{O}_1 \Subset \mathscr{O}_2$, *viz.* the closure of $\mathscr{O}_1$ is contained in the interior of $\mathscr{O}_2$, there exists a vector $\Omega_{\mathscr{O}_1,\mathscr{O}_2} \in \mathscr{H}_0$ such that for all operators $R_1 \in \pi_0(\mathfrak{A}(\mathscr{O}_1))$ and $R_2' \in \pi_0(\mathfrak{A}(\mathscr{O}_2))'$ one has

$$\langle \Omega_{\mathscr{O}_1,\mathscr{O}_2}, R_1 R_2' \, \Omega_{\mathscr{O}_1,\mathscr{O}_2} \rangle = \langle \Omega_0, R_1 \Omega_0 \rangle \langle \Omega_0, R_2' \Omega_0 \rangle . \tag{7.2}$$

Here the prime $'$ indicates the algebra of all bounded operators on $\mathscr{H}_0$ which commute with the given algebra. By locality, $\pi_0(\mathfrak{A}(\mathscr{O}_2))' \supset \pi_0(\mathfrak{A}(\mathscr{O}_2^c)$, so the equality shows that there exist product states in the vacuum sector in which measurements in a bounded region and its spacelike separated complement are completely uncorrelated. One then considers the projection $E_{\mathscr{O}_1,\mathscr{O}_2}$ onto the subspace of $\mathscr{H}_0$ spanned by

$$R_1 \, \Omega_{\mathscr{O}_1,\mathscr{O}_2} , \quad R_1 \in \pi_0(\mathfrak{A}(\mathscr{O}_1)) . \tag{7.3}$$

This projection commutes with all elements of $\pi_0(\mathfrak{A}(\mathscr{O}_1))$ and transfers the elements of $R_2' \in \pi_0(\mathfrak{A}(\mathscr{O}_2))'$ into the vacuum state,

$$E_{\mathscr{O}_1,\mathscr{O}_2} R_2' E_{\mathscr{O}_1,\mathscr{O}_2} = \langle \Omega_0, R_2' \Omega_0 \rangle \, E_{\mathscr{O}_1,\mathscr{O}_2} \,, \quad R_2' \in \pi_0(\mathfrak{A}(\mathscr{O}_2))' . \tag{7.4}$$

As an aside, if one replaces in (7.3) the algebra $\pi_0(\mathfrak{A}(\mathscr{O}_1))$ by $\pi_0(\mathfrak{A}(\mathscr{O}_2))'$, one arrives at a projection in $\pi_0(\mathfrak{A}(\mathscr{O}_2))^-$ that transfers the elements of $\pi_0(\mathfrak{A}(\mathscr{O}_1))$ into the vacuum state. This entails an algebraic version of the split property. Turning back to the problem at hand, one finds that the operators $E_{\mathscr{O}_1,\mathscr{O}_2} e^{-(1/T)H} E_{\mathscr{O}_1,\mathscr{O}_2}$ have a finite trace on $\mathscr{H}_0$. Dividing the operators by its value, one obtains density matrices which replace in the present setting the Gibbs-von Neumann ensembles. Moreover, for suitably increasing regions $\mathscr{O}_1 \Subset \mathscr{O}_2$ approaching $\mathscr{M}$, the resulting sequences of states have weak limits on the algebra $\mathfrak{A}$, which are designed to describe global equilibrium states for the given temperature T.

As was shown by Haag, Hugenholtz, and Winnink, it is a distinctive property of equilibrium states at a given temperature that they satisfy the KMS condition, which is briefly recalled here. Let $\alpha(t)$, $t \in \mathbb{R}$, be the automorphism inducing the time translations on $\mathfrak{A}$ in a chosen Lorentz frame. A state ω_T on $\mathfrak{A}$ satisfies the KMS condition at temperature T if each correlation function

$$t \mapsto \omega_T(A_1 \alpha(t)(A_2)) \,, \quad A_1, A_2 \in \mathfrak{A} \,, \tag{7.5}$$

can be extended to the strip $S_T := \{z \in \mathbb{C} : 0 \leq \mathrm{Im} z \leq (1/T)\}$, is continuous there and analytic in the interior, and has at the upper rim the boundary value $t \mapsto \omega_T(\alpha(t)(A_2)A_1)$. Any such state is invariant under the action of the time translations. The thermodynamic limit states described above comply with this condition if $\mathfrak{A}$ consists of operators which transform norm-continuously under the action of the time evolution. It is an open problem whether this technical assumption can be dropped.

The discovery of the KMS condition in algebraic quantum field theory provided a fruitful link with developments in the theory of operator algebras, *viz.* modular theory, established by Tomita and Takesaki. We briefly summarize here some essential points, cf. also [46, E3]. Within the mathematical setting, one deals with weakly closed operator algebras $\mathfrak{R}$ which are represented on some Hilbert space $\mathscr{H}$ and have a cyclic and separating vector Ω; that is, the subspace $\mathfrak{R}\Omega$ is dense in $\mathscr{H}$ and, apart from 0, there is no operator in $\mathfrak{R}$ which annihilates Ω. Given these ingredients, one considers the antilinear operator $S : \mathfrak{R}\Omega \to \mathscr{H}$ given by

$$S R \Omega := R^* \Omega \,, \quad R \in \mathfrak{R} \,. \tag{7.6}$$

It is a densely defined, closable operator whose closure has a polar decomposition, denoted by $S = J\Delta^{1/2}$. Here the operator J, called modular conjugation, is an antiunitary involution, $J^2 = 1$; the operator Δ, called modular operator, is positive and satisfies $\Delta\Omega = \Omega$. The central result obtained in this setting are the equalities of sets, involving the adjoint actions of these operators on the algebra $\mathfrak{R}$,

$$J\mathfrak{R}J = \mathfrak{R}', \qquad \Delta^{it}\,\mathfrak{R}\Delta^{-it} = \mathfrak{R},\ t \in \mathbb{R}. \tag{7.7}$$

Denoting by $\delta(s)$ the adjoint action of the unitaries Δ^{is} on $\mathfrak{R}$, $s \in \mathbb{R}$, called modular automorphism group, one considers the correlation functions

$$s \mapsto \langle \Omega, R_1 \delta(s)(R_2)\Omega \rangle, \quad R_1, R_2 \in \mathfrak{R}. \tag{7.8}$$

Remarkably, they satisfy the KMS condition for temperature 1; it can be replaced by any other value by an adjustment of the exponent of the modular operator. There is an important converse of this result with regard to its applications in physics: in the GNS-representation induced by a KMS state on $\mathfrak{A}$, the dynamics coincides (up to rescalings) with the corresponding modular automorphism group, acting on the weak closure of the represented algebra of observables.

These observations have found numerous applications in AQFT, where the occurrence of cyclic and separating vectors for (sub)algebras of the observables $\mathfrak{A}$ is quite common (Reeh-Schlieder property [41]). With regard to the physical interpretation of the theory, a prominent result is the Bisognano-Wichmann theorem which deals with the vacuum representation of the subalgebras of observables localized in regions bounded by two lightlike planes (wedges). It says that the modular group for any such wedge algebra and the vacuum state coincides with the automorphic action of the one-parameter group of boosts leaving the wedge invariant. Since the boosts can be interpreted as dynamics of a uniformly accelerated observer, this result establishes the general nature of the Unruh effect in AQFT [43]. Moreover, the respective modular conjugations agree with the PCT-operator, multiplied with a Poincaré transformation which transforms the parity P into a reflection about the spatial plane tangent to the wedge.

The discovery that in this particular case the modular structure is related to spacetime symmetries has triggered interest in the modular operators of other regions, such as double cones or lightcones. Indeed, if the underlying theory has a sufficiently big symmetry group, such as in conformal field theory, the modular groups of double cone or lightcone algebras and the vacuum state act like specific subgroups of the conformal group on these algebras, cf. for example [37]. In general, however, such a geometric behaviour may not be expected. Even in case of a non-interacting massive scalar field the specific properties of the modular groups for double cones and the vacuum state are not yet known.

Modular theory has also contributed to progress in the structural analysis of AQFT. It is an important consequence of the Bisognano-Wichmann theorem [3] that a version of the condition of Haag duality (essential duality) is satisfied in the vacuum sector. This feature is a vital ingredient in sector analysis. Moreover, with the help of modular theory, the Murray-von Neumann type was determined of the weak closures of algebras of observables in various regions. These algebras were shown to be of type III_1 irrespective of the underlying theory [29]. This feature is a consequence of the sharp boundaries of the regions so that observations in the interior and exterior are strongly entangled. It results in algebraic properties which are quite different from those known of the algebra of all bounded operators on a Hilbert space. In particular, there exists no meaningful trace on the local algebras. This complicates the implementation of basic physical concepts, such as entropy. One has either to rely on the existence of algebras, admitting a trace, that contain a given local algebra and which are themselves contained in a slightly larger local algebra. This corresponds to sealing off a laboratory from the outside with walls (split property [25]). Or one may enlarge the local algebras by proceeding to the crossed product with their modular groups. The resulting von Neumann algebras are known to admit a (not necessarily unique) trace, they are of type II. It was recently recognized that these enlargements can be interpreted as couplings of the local algebras through the modular groups with the reference systems of observers [22, 27].

Another area of applications of modular theory is the analysis of the degrees of freedom of field theoretic models. The maps introduced in relation (7.1), based on Hamiltonians, can be replaced by similar maps involving the modular operators. It led to the formulation of modular nuclearity conditions which are less restrictive [14]. But they still imply the existence of product states and projections in the vacuum sector, cf. equation (7.4). These observations allowed the construction of a large family of integrable models within the algebraic framework, which were not accessible by other constructive means [E4]. Thus the results of an extensive structural analyses in the general framework of AQFT provided the ground for these novel constructive methods.

8 Further topics

There exist key results which, originally, were derived in the Wightman framework of quantum field theory, such as the spin-statistics theorem and the PCT theorem [44]. Using the Doplicher-Haag-Roberts theory of superselection sectors and its later developments, these results were established in AQFT in much greater generality [16, 23, 24]. For example, the Bose-Fermi alternative of statistics was derived and not simply assumed, such as in the Wightman framework. Moreover, the spin-statistics theorem was established in massive theories for particles carrying a non-localizable charge, which do not fit into the Wightman framework. In all of these theories, the existence of charge conjugate sectors was established, along with their finite statistics. In contrast, these features had to be postulated in the Wightman framework. Finally, in low dimensional spacetimes the appearance of braid group statistics leads to modifications of these results which were established in AQFT as well [E4, E9].

Point-localized quantum fields do not occur explicitly in the framework of AQFT. But their existence was established under appropriate conditions [30], restricting, for example, the size of the phase space of the theory [6]. These fields can be recovered from the local algebras as operator valued distributions which transform covariantly under Poincaré transformations. Moreover, there exist algebraic relations between the pointlike fields (operator product expansions) that encode characteristic features of a theory [7].

Such pointlike structures play also a crucial role in AQFT on globally hyperbolic spacetimes, where spacetime symmetries are generically absent. So they cannot be used for the identification of observables in different regions. Instead, one relies on a more fundamental principle of general covariance [11]. In case of symmetric spacetimes, one recovers from this principle the automorphic action of the symmetries on observables. For general spacetimes it implies that point fields, such as the energy momentum tensor, can be used in order to identify the observables in different regions. This fact is, for example, an important ingredient in the perturbative renormalization of quantum field theories on curved spacetimes [38, 39].

References

[E] The references marked with an E in front of their numbers are articles solicited for this encyclopedia. The titles may change.

[E1] Axiomatic quantum field theory

[E2] Thermal quantum field theory

[E3] Tomita-Takesaki modular theory

[E4] Construction of two-dimensional models in algebraic quantum field theory

[E5] Perturbative algebraic quantum field theory

[E6] Triviality of ϕ_4^4

[E7] Entanglement entropy in quantum field theory

[E8] Scattering in relativistic quantum field theory: basic concepts, tools, and results

[E9] Infrared problem in quantum field theory

[E10] Quantum field theory on curved spacetimes

[E11] Symmetries in quantum field theory: superselection sectors

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