An Algebraic Approach to Quantum Field Theory

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It is shown that two quantum theories dealing, respectively, in the Hilbert spaces of state vectors $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are physically equivalent whenever we have a faithful representation of the same abstract algebra of observables in both spaces, no matter whether the representations are unitarily equivalent or not. This allows a purely algebraic formulation of the theory. The framework of an algebraic version of quantum field theory is discussed and compared to the customary operator approach. It is pointed out that one reason (and possibly the only one) for the existence of unitarily inequivalent faithful, irreducible representations in quantum field theory is the (physically irrelevant) behavior of the states with respect to observations made infinitely far away. The separation between such "global" features and the local ones is studied. An application of this point of view to superselection rules shows that, for example, in electrodynamics the Hilbert space of states with charge zero carries already all the relevant physical information.

I. INTRODUCTION

THE essential feature which distinguishes quantum field theory within the frame of general quantum physics is the principle of locality. This principle states that it is meaningful to talk of observables which can be measured in a specific space-time region and that observables in causally disjoint regions are always compatible. It is then natural to introduce the following concepts: If **B** is a region in Minkowski space, we denote by $\mathfrak{A}(\mathbf{B})$ the algebra generated by the observables in **B**. A specific field theory will fix the correspondence between regions and algebras

$$\mathbf{B} \rightarrow \mathfrak{A}(\mathbf{B}). \qquad (1)$$

In fact, we may consider this correspondence to be the content of the theory. Indeed, once it is known, one can calculate quantities of direct physical interest such as masses of particles and collision cross sections.[1]

This approach has been developed in previous work[2,3] within the customary framework of quantum theory in which observables are considered to be (bounded or unbounded) operators on a Hilbert space. The algebras $\mathfrak{A}(\mathbf{B})$ are then concrete *-algebras of operators and it is mathematically convenient to replace $\mathfrak{A}(\mathbf{B})$ by its associated von Neumann ring $\mathbf{R}(\mathbf{B})$. Properties of this family of von Neumann rings, which follow from general physical principles or are suggested by conventional quantum field theory, have been studied in Ref. 2. Particle aspects and collision theory are treated in Ref. 3.

In the present paper we shall be concerned with another question. Suppose that the algebras $\mathfrak{A}(\mathbf{B})$ are abstractly defined (without reference to operators on a Hilbert space).[4] If we consider a faithful realization of the algebraic elements by operators on a Hilbert space we come back to the previous point of view. However, we expect that there are many unitarily inequivalent irreducible representations. This ambiguity, typical of quantum field theory, has been the subject of some discussion within the past decade.[5] To deal with it, most authors assume that there is one and only one representation space in which the physical vacuum state appears as a vector and that we have to single out this particular representation as the physically

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[1] At the present stage this claim is an overstatement, but it is a reasonable extrapolation of results described in Ref. 3.

[2] R. Haag, *Colloque Internationale sur les Problèmes Mathématiques de la Théorie Quantique des Champs, Lille, 1957* (Centre National de la Recherche Scientifique, Paris, 1958); R. Haag and B. Schroer, J. Math. Phys. **3**, 248 (1962); H. Araki, "Einführung in die Axiomatische Quantenfeldtheorie," Lecture notes at the Eidgenössischen Technischen Hochschule, Zürich, 1961/62, unpublished.

[3] R. Haag, Phys. Rev. **112**, 669 (1958); D. Ruelle, Helv. Phys. Acta **35**, 147 (1962); H. Araki, see Ref. 2.

[4] In a heuristic manner the commutation relations and field equations of a conventional quantum field theory provide such an abstract characterization.

[5] It was first noticed in the example of various algebras associated with infinitely many creation and destruction operators. See J. von Neumann, Comp. Math. **6**, 1 (1938); K. O. Friedrichs, *Mathematical Aspects of the Quantum Theory of Fields* (Interscience Publishers, Inc., New York, 1953). For further discussions of this phenomenon in its relation to various models in quantum field theory see, for instance, L. Van Hove, Physica **18**, 145 (1952); A. S. Wightman and S. S. Schweber, Phys. Rev. **98**, 812 (1955); R. Haag, Kgl. Danske Videnskab. Selskab Mat.-Fiz. Medd. **29**, No. 12 (1955); I. E. Segal, Trans. Am. Math. Soc. **88**, 12 (1958); J. Lew, Ph.D. thesis, Princeton Univ., 1960, unpublished; and the papers cited in Ref. 6.

relevant one.[6] While this attitude appears to be perfectly consistent it does not go to the heart of the matter. The fact that no actual measurement can be performed with absolute precision implies that the realistic notion of "physical equivalence" is far less stringent than that of unitary equivalence. Our discussion in Sec. II shows that this notion coincides with the mathematical concept of "weak equivalence" as introduced by Fell.[7]

Fell's results then imply that all faithful representations are in fact physically equivalent, thus opening the way to a purely algebraic approach to the theory. The distinction between "physical" and "unitary" equivalence was forced on us by the discussion of examples in a recent paper.[8]

The purely algebraic approach to the theory has been championed for many years by Segal.[9] He pointed out that many questions of physical interest (e.g., the determination of spectral values) can be answered without reference to a Hilbert space if one chooses the algebra of observables to be a C^*-algebra.[10] Applying these ideas to quantum field theory Segal expected to circumvent the difficulties associated with the existence of inequivalent representations.[11] So far this approach has stayed, however, in a somewhat experimental stage, i.e., it has not yet led to a well-defined frame in which a satisfactory physical interpretation is specified. It is the purpose of this paper to establish such a frame making essential use of the principle of locality. This frame is very similar to that in Ref. 2 but differs from it in two respects. First, we consider the algebras $\mathfrak{A}(\mathbf{B})$ as (abstract) C^*-algebras, not as operator algebras on a Hilbert space. Secondly, we exclude from the list of "all" observables those quantities which refer to infinitely extended regions. Thus the total energy, total charge, etc., are considered as unobservable. This is of particular importance in connection with superselection rules (see Sec. III).

We turn now to a precise specification of the frame:

(1) The "regions" $\mathbf{B}$ for which the correspondence (1) is defined shall be the open sets with compact closure[12] in Minkowski space, the algebras $\mathfrak{A}(\mathbf{B})$ shall be (abstract) C^*-algebras.

(2) Isotony: If $\mathbf{B}_1 \supset \mathbf{B}_2$ then $\mathfrak{A}(\mathbf{B}_1) \supset \mathfrak{A}(\mathbf{B}_2)$. We assume in addition that one of the two following situations prevails. Either $\mathfrak{A}(\mathbf{B}_1)$ and $\mathfrak{A}(\mathbf{B}_2)$ have a common unit element, or neither of them has a unit. The first situation can be obtained from the second by formal adjunction of a unit.

(3) Local Commutativity: If $\mathbf{B}_1$ and $\mathbf{B}_2$ are completely spacelike with respect to each other, then $\mathfrak{A}(\mathbf{B}_1)$ and $\mathfrak{A}(\mathbf{B}_2)$ commute.

(4) The set-theoretic union of all $\mathfrak{A}(\mathbf{B})$ is a normed *-algebra.[13] Taking its completion we get a C^*-algebra which we denote by $\mathfrak{A}$ and call the *algebra of quasilocal observables*. We maintain that $\mathfrak{A}$ contains all observables of interest.[14]

(5) Lorentz Covariance: The inhomogeneous Lorentz group is represented by automorphisms $A \in \mathfrak{A} \to A^L \in \mathfrak{A}$ such that

$$\mathfrak{A}(\mathbf{B})^L = \mathfrak{A}(L\mathbf{B}), \tag{2}$$

where $L\mathbf{B}$ is the image of the region $\mathbf{B}$ under the Lorentz transformation L.

(6) $\mathfrak{A}$ is primitive (see Appendix).

Concerning the physical interpretation the essential point is, of course, that the algebra of observables $\mathfrak{A}$ has a texture, namely the family of subalgebras $\mathfrak{A}(\mathbf{B})$, and that the elements of $\mathfrak{A}(\mathbf{B})$ are interpreted as representing physical operations performed in the region $\mathbf{B}$. In Sec. II we discuss to some extent how this information can be exploited and we justify the previously mentioned notion of physical equivalence of representations. Section III deals with the separation of global and local aspects and its application to superselection rules. Section IV gives a brief comparison between the present approach and the operator approach.

II. PHYSICAL INTERPRETATION OF AN ALGEBRAIC SCHEME AND PHYSICAL EQUIVALENCE OF REPRESENTATIONS

We are concerned with two categories of objects: "states" and "operations." The term "state" is

[6] In Wightman's approach the existence of a vacuum state and the relevant properties of this state are postulated on physical grounds. See, e.g., A. S. Wightman, Phys. Rev. **101**, 860 (1956). The following papers discuss the existence and uniqueness of a vacuum state for specific models. H. Araki, J. Math. Phys. **1**, 492 (1960); D. Shale, Ph.D. Thesis, Department of Mathematics, University of Chicago, 1961, unpublished; I. E. Segal, Illinois J. Math. **6**, 500 (1962); H. J. Borchers, R. Haag, and B. Schroer, Nuovo Cimento **29**, 148 (1963).

[7] J. M. G. Fell, Trans. Am. Math. Soc. **94**, 365 (1960).

[8] H. J. Borchers, R. Haag, and B. Schroer, see Ref. 6.

[9] I. E. Segal, Ann. Math. **48**, 930 (1947).

[10] For definitions and relevant theorems see Appendix 1.

[11] I. E. Segal, *Colloque Internationale sur les Problèmes de la Théorie Quantique des Champs, Lille, 1957* (Centre National de la Recherche Scientifique, Paris, 1958).

[12] Physically speaking: 4-dimensional regions with finite extension.

[13] The union of all $\mathfrak{A}(\mathbf{B})$ has an obvious *-algebra structure due to the isotony assumption. Furthermore, the norm of one of its elements is the same in all local algebras, $\mathfrak{A}(\mathbf{B})$ containing it due to the uniqueness of the C^*-norm (see Appendix 1).

[14] $\mathfrak{A}$ is the collection of the uniform limits of all (bounded) observables describing measurements performable in finite regions of space-time. By taking uniform limits we do not essentially change the local character of the observables (hence the name quasilocal).

used for a statistical ensemble of physical systems,[15] the term "operation" for a physical apparatus which may act on the systems of an ensemble during a limited amount of time producing a transformation from an initial state to a final state. We assume that any operation is applicable to any state. This is one of the idealizations inherent in quantum physics. One is frequently interested in operations which transmit only a certain fraction of the systems of the initial ensemble and eliminate the others. This fraction (probability) is a number depending on the initial state and on the operation. It is the one piece of information about the state which is gathered by the experimenter performing the operation.[16] An experiment may always be regarded as the determination of the transmission probabilities for a finite number of operations.

We may say therefore that we have a complete theory if we are able in principle to compute such probabilities for every state and every operation when the state and the operation are defined in terms of laboratory procedures.

It is not the objective of this paper to justify the particular mathematical formalism by means of which "states" and "operations" are represented in quantum theory.[17] We accept here uncritically the following formal structure:[18] One has an algebra $\mathfrak{A}$ (which in our case will be identified with the C^*-algebra described in the introduction). A "state" is mathematically represented by a positive linear form (expectation functional) over $\mathfrak{A}$. Explicitly, if φ denotes a state, then for every $A \in \mathfrak{A}$ we have a complex number $\varphi(A)$, depending linearly on A and such that

$$\varphi(A^*A) \geq 0. \quad (3)$$

The value which φ takes for the unit element I of the algebra defines the normalization of the state. Intuitively speaking, $\varphi(I)$ is proportional to the number of systems of which the ensemble is composed (the proportionality factor being irrelevant). One calls $\varphi(I)$ the norm of the state φ.[19] The collection of all positive linear functionals over $\mathfrak{A}$ is the positive cone of the dual space $\mathfrak{A}^*$ of the algebra and is therefore denoted by $\mathfrak{A}^{*(+)}$. A functional $\varphi \in \mathfrak{A}^{*(+)}$ is called "extremal" if it cannot be decomposed into a positive linear combination of two others, i.e., if $\varphi = \alpha\varphi_1 + \beta\varphi_2$ with $\alpha > 0$, $\beta > 0$, $\varphi_1 \in \mathfrak{A}^{*(+)}$, $\varphi_2 \in \mathfrak{A}^{*(+)}$ is impossible except for the trivial solution $\varphi_1 = \lambda\varphi$. The extremal functionals correspond to pure states.

An "operation" is mathematically represented by a linear transformation of $\mathfrak{A}^*$ which maps $\mathfrak{A}^{*(+)}$ into itself and does not increase the norm. Those special operations which transform pure states into pure states are called "pure operations." It is asserted that the pure operations are in one to one correspondence with the elements contained in the unit sphere of the algebra (elements $A \in \mathfrak{A}$ with $||A|| \leq 1$).[20] The transformation of the (general) state φ by the pure operation A is given by $\varphi \to \varphi_A$ with

$$\varphi_A(B) = \varphi(A^*BA). \quad (4)$$

Therefore one gets for the transmission probability of φ through A the expression

$$P(\varphi, A) = \varphi(A^*A)/\varphi(I). \quad (5)$$

Apart from the emphasis on "operations" instead of "observables" the preceding paragraph was just a description of the standard formal structure of quantum physics. It may be useful to point out the difference between the Hilbert space approach[21] and the purely algebraic approach[22] as far as this general formalism is concerned. In the former case the only states considered are the density matrices in the representation space (positive-definite self-adjoint operators with finite trace). This collection of states is a subset (usually not the whole) of $\mathfrak{A}^{*(+)}$. The purely algebraic approach on the other hand considers all elements of $\mathfrak{A}^{*(+)}$ as possible states. One has to ask therefore whether this richer supply of states makes the physical interpretation more difficult.

We must turn now to the physical interpretation, i.e., to the following question: Suppose a specific

[15] We adopt Segal's terminology in which the word state is used for any statistical ensemble. If the ensemble cannot be decomposed into purer ones it is called a "pure state," otherwise an "impure state" ("mixture" in von Neumann's terminology).

[16] We find it preferable to base our discussion on the notion of "operations" as defined above instead of "observables" as used by Dirac and von Neumann. An "observable" in the technical sense is an idealization, which in general implies suitably defined limits of an infinite number of operations. It is thus a far less simple concept.

[17] We hope to discuss this question in another paper.

[18] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, 1955); I. E. Segal, see Ref. 9.

[19] It is not crucial to assume that the algebra contains the unit element; see Appendix 1. The norm is then defined as $\sup |\Phi(A)|/||A||$.

[20] It has been emphasized by H. Ekstein that, in general, one algebraic element will correspond to many different laboratory procedures which are equivalent insofar as they produce the same transformation of the states. For simplicity we shall, however, always speak of an "operation" instead of an "equivalence class of operations."

[21] J. von Neumann, see Ref. 18.

[22] I. E. Segal, see Ref. 9.

operation (or state) is defined in terms of a laboratory procedure. How do we find the corresponding element in the mathematical description? For the "operations" the question is partially answered by the assertion: An operation in the space–time region **B** corresponds to an element from $\mathfrak{A}(\mathbf{B})$. It is very likely that this simple statement provides not only a partial answer but a complete one because ultimately all physical processes are analyzed in terms of geometric relations of (unresolved) phenomena. In any case it is rather evident that one can construct a good mathematical representative of a Geiger counter coincidence arrangement using the subalgebras for finite regions.[23]

The remaining task is to bridge the gap between the physical and the mathematical description of a state. One possible attitude is, of course, to say that the state φ may be described physically (as well as mathematically) by the collection of probabilities $P(\varphi, A)$ for all pure operations A. This would mean that, once an ensemble has been prepared, "somehow" we make all sorts of monitoring experiments to find out which ensemble we have. The other attitude is to assume that the ensemble is prepared by means of a single, specific operation from an initial ensemble which is completely unknown. In practice both methods (preparation of ensemble by a "filtering" operation from an unknown ensemble and determination of ensemble by monitoring experiments) are used to supplement each other. In both respects it is clear that the physical interpretation of *states* is fixed once we know the correspondence between the mathematical and the physical description for the *operations*. It is, however, also clear that no actual experiment will enable us to establish a definite state.

Take, first, the case of monitoring experiments on an ensemble. These will provide us with a finite number of probabilities, measured with a finite accuracy. In the mathematical scheme this information characterizes exactly a neighborhood in the weak topology of $\mathfrak{A}^*$. Namely, if the probabilities are the transmission probabilities for a collection of pure operations A_i $(i = 1, \cdots, N)$ then we know that the state satisfies[24]

$$\varphi(I) = 1, \tag{6}$$

$$|\varphi(A_i^* A_i) - p_i| < \epsilon_i, \tag{7}$$

where p_i are the experimentally determined probabilities and ϵ_i the errors. If, alternatively, one wants to define the ensemble in terms of a single preparatory operation then the discussion is mathematically somewhat more difficult. Let T be the preparing operation and $\mathbf{R}_T$ its range (the image of $\mathfrak{A}^{*(+)}$ under T). Then the only certain knowledge about the prepared state is that it lies in $\mathbf{R}_T$. To obtain definite state we need an operation with a one-dimensional range. There are two reasons why such an ideal operation is impossible. The first has to do with the limited accuracy in the specification of T [the counterpart of the errors ϵ_i in Eq. (7)]. The other comes from the special structure of our algebra $\mathfrak{A}$. Namely, it is evident that no quasilocal operation can have a finite-dimensional range because an operation in a finite region has no effect on the physical situation in a causally disjoint region. While we are unable at the moment to give a precise analysis of the consequences of these two limitations we feel that the first one (limitation in accuracy) will result in the statement that we have in no actual experiment a precisely specified state but rather a weak neighborhood in $\mathfrak{A}^{*(+)}$. This is the conclusion relevant to the remaining discussion in this section. The other limitation, arising from the special nature of the quasilocal algebra probably implies that there exist many unitarily inequivalent irreducible representations of $\mathfrak{A}$. (See Sec. III and Ref. 5).

The foregoing discussion leads us now to the following:

Statement. Let $R^{(1)}$ and $R^{(2)}$ be two representations of $\mathfrak{A}$ and Ω_1, Ω_2 the subsets of states which correspond to density matrices in the two representation spaces. The two representations are *physically equivalent* if every weak neighborhood of any element of Ω_1 contains an element of Ω_2 and vice versa.

The notion of physical equivalence coincides exactly with that of "weak equivalence" as defined by Fell.[25] We can apply *Fell's equivalence theorem*, i.e., two representations are weakly equivalent if and only if they have the same kernel.[26]

The conclusion is thus that all faithful representations of $\mathfrak{A}$ are physically equivalent. The relevant object is the abstract algebra and not the representation. The selection of a particular (faithful) representation is a matter of convenience without physical

[23] This will be discussed in a separate paper. See also Ref. 3.

[24] We chose (arbitrarily) the normalization of the state. For the sake of symmetry with Eq. (7) we could write, instead of (6), equally well,

$$|\Phi(I) - 1| < \epsilon_0.$$

[25] See Ref. 7 and the last paragraph of Appendix I.

[26] The kernel of a representation is the collection of all elements of $\mathfrak{A}$ which are represented by zero.

implications. It may provide a more or less handy analytical apparatus.

It also follows that we should consider only faithful representations because, supposing for a moment that a nonfaithful representation with kernel $\mathbf{K}$ contained all physically relevant information then the only physically equivalent representations would be those with the same kernel. The relevant object is then not the algebra $\mathfrak{A}$ but the quotient $\mathfrak{A}/\mathbf{K}$. According to a well-known theorem this quotient is again a C^*-algebra, and we should have taken this algebra in the first place instead of $\mathfrak{A}$.

As a final remark we might add that it appears natural to assume that $\mathfrak{A}$ is primitive, i.e., that it has at least one representation which is both faithful and irreducible. It would be tempting to assume even that $\mathfrak{A}$ is simple, i.e., that all its representations are faithful.[27]

III. LOCAL AND GLOBAL PROPERTIES. SUPERSELECTION RULES

It was pointed out in the introduction that all actual experiments involve only operations in finite space-time regions. Hence it is natural to introduce the notion of a "partial state with respect to a region."

Definition. A partial state with respect to region $\mathbf{B}$ is a positive linear form over the algebra $\mathfrak{A}(\mathbf{B})$ or, alternatively speaking, an equivalence class of "global states" (positive linear forms over $\mathfrak{A}$) which coincide on $\mathfrak{A}(\mathbf{B})$.

The two alternative definitions are equivalent due to the following theorem which we use again later:

Theorem.[28] If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are two C^*-algebras and $\mathfrak{A}_1 \subset \mathfrak{A}_2$ then every state (positive linear form) over $\mathfrak{A}_1$ can be extended to at least one state over $\mathfrak{A}_2$. A pure state over $\mathfrak{A}_1$ can be extended to a pure state over $\mathfrak{A}_2$.

It is of interest to understand the coupling between the partial states of different regions which results from algebraic relations between the various subalgebras $\mathfrak{A}(\mathbf{B})$. We shall call the partial states in regions $\mathbf{B}_1$ and $\mathbf{B}_2$ completely uncoupled if, choosing an arbitrary pair of equally normalized partial states $\varphi^{(1)} \in \mathfrak{A}(\mathbf{B}_1)^{*(+)}$ and $\varphi^{(2)} \in \mathfrak{A}(\mathbf{B}_2)^{*(+)}$, one can find a global state φ which is an extension of both $\varphi^{(1)}$ and $\varphi^{(2)}$. The extreme opposite of this situation (i.e., complete coupling) prevails if each partial state in $\mathbf{B}_1$ determines uniquely a partial state in $\mathbf{B}_2$ by the process of extension to $\mathfrak{A}$ and restriction to $\mathfrak{A}(\mathbf{B}_2)$.

On physical grounds we want:

(i) If $\mathbf{B}_2$ is contained in the causal shadow of $\mathbf{B}_1$ then the partial states in $\mathbf{B}_2$ are uniquely determined by those in $\mathbf{B}_1$ (causality).

(ii) If $\mathbf{B}_1$ and $\mathbf{B}_2$ are causally disjoint then the partial states in the two regions are essentially[29] uncoupled (locality).

Property (i) is equivalent to the algebraic requirement

$$\mathfrak{A}(\mathbf{B}_2) \subset \mathfrak{A}(\mathbf{B}_1) \tag{8}$$

for all regions $\mathbf{B}_2$ in the causal shadow of $\mathbf{B}_1$.[30] Property (ii) is related to the local commutativity postulate but we do not know whether this postulate is already enough to guarantee the lack of coupling for partial states in causally disjoint regions or whether some further structure property is needed.

We now give a brief intuitive (nonrigorous) discussion of some phenomena for which the distinction between global and local features plays a role.

A. Existence of Unitarily Inequivalent Irreducible Representations of $\mathfrak{A}$

A pure state over $\mathfrak{A}$ corresponds to a vector in some irreducible representation space of $\mathfrak{A}$. Two pure states belong to the same representation if and only if the one results from the other by transformation with an element of the algebra [in the sense of Eq. (4)].[31] Otherwise they belong to representations which are unitarily inequivalent.

We confine our attention to the states without infinitely extended correlations. These states are characterized by the property that an operation in region $\mathbf{B}$ does not affect the partial state in a far

[27] Compare B. Misra, "On the algebra of quasi-local operators of Quantum Field Theory," to be published. See, however, Appendix II for an example of a nonsimple algebra of physical interest.

[28] See, e.g., M. A. Neumark, Ref. 42.

[29] From the physical point of view it would not be necessary that the uncoupling is complete if the separation distance between $\mathbf{B}_1$ and $\mathbf{B}_2$ is finite but only that it becomes complete in the limit of infinite spacelike separation.

[30] Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two subalgebras of $\mathfrak{A}$ and $\mathfrak{A}_1^{\perp}$, $\mathfrak{A}_2^{\perp}$ those subspaces of $\mathfrak{A}^*$ which are composed of the linear forms vanishing respectively on $\mathfrak{A}_1$ and $\mathfrak{A}_2$. If the "partial states" over $\mathfrak{A}_1$ determine those over $\mathfrak{A}_2$ we have

$$\mathfrak{A}_2^{\perp} \supset \mathfrak{A}_1^{\perp}.$$

Thus

$$\mathfrak{A}_2^{\perp\perp} \subset \mathfrak{A}_1^{\perp\perp}.$$

But $\mathfrak{A}_i^{\perp\perp}$, considered as a subset of $\mathfrak{A}$, coincides with the uniform closure of $\mathfrak{A}_i$, i.e., with $\mathfrak{A}_i$ itself.

[31] Compare Appendix I, Kadison's Theorem, Ref. 53.

away region $\mathbf{B}'$ (apart from a change in normalization).[32] In symbols:

$$\varphi(QQ') \approx \varphi(Q)\varphi(Q')/\varphi(I) \tag{9}$$

if Q and Q' belong to the algebras of two far-separated regions.

Consider now an infinite collection of causally disjoint regions $\mathbf{B}_k$. Let Φ be a pure state without infinitely extended correlations; the corresponding partial state in $\mathbf{B}_k$ will be denoted by φ_k. It is clear that a transformation of Φ by an element from $\mathfrak{A}$ will not change the asymptotic tail ($k \to \infty$) of the sequence of partial states φ_k, apart from a common normalization factor, because any element of $\mathfrak{A}$ can be approximated to arbitrary precision by an operation in a finite region. The asymptotic tail of the sequence of partial states φ_k is thus common to all normalized states belonging to the same representation as Φ. It is a "unitary invariant". On the other hand the lack of coupling between partial states in causally disjoint regions [property (ii)] suggests that there is an enormous variety of possible asymptotically different sequences φ_k. This gives us then many unitarily inequivalent representations of $\mathfrak{A}$. They differ in the global aspects of their states but this difference is irrelevant as long as we are interested only in experiments in finite regions ("physical equivalence" of all these representations).

B. Lorentz Transformations

As postulated under item (5) in the introduction we have for every element L of the inhomogeneous Lorentz group an automorphism of the algebra: $A \to A^L$. The automorphism defines a corresponding transformation in the state space, namely

$$\varphi \to \varphi_L \quad \text{with} \quad \varphi_L(A) = \varphi(A^L). \tag{10}$$

This is a linear transformation in the Banach space $\mathfrak{A}^*$ which preserves the norm and transforms the positive cone $\mathfrak{A}^{*(+)}$ into itself. Since

$$(\varphi_{L_1})_{L_2} = \varphi_{L_2L_1}, \tag{11}$$

we get in this way an "antirepresentation" of the Lorentz group (by isometric operators in a Banach space). In the special case when L is a translation by a 4-vector x we write A^x and φ_x for the transformed quantities.

Since Lorentz transformations are global operations, affecting the far away regions as strongly (or even stronger) than the regions nearby, they do not correspond to elements in the quasilocal algebra $\mathfrak{A}$. Indeed, if we assumed that there is an element $U(L)$ in $\mathfrak{A}$ such that

$$A^L = U(L)AU^{-1}(L) \quad \text{for all} \quad A \in \mathfrak{A}, \tag{12}$$

we would get an immediate contradiction.[33] We could then approximate $U(L)$ by an operation C in a finite region $\mathbf{B}$ such that $||U(L) - C|| < \epsilon$. Taking $A \in \mathfrak{A}(\mathbf{B}')$ with $\mathbf{B}'$ causally disjoint from $\mathbf{B}$, Eq. (12) would imply

$$||A^L - A|| < 2\,||A||\,\epsilon \quad \text{for all} \quad A \in \mathfrak{A}(\mathbf{B}'),$$

which is *not* true.

We may assume, however, that there are elements in $\mathfrak{A}$ which produce the same effect as a Lorentz transformation within an arbitrarily chosen finite region $\mathbf{B}$. Denoting such an element by $U_{\mathbf{B}}(L)$ we have instead of (12)

$$A^L = U_{\mathbf{B}}(L)AU_{\mathbf{B}}^{-1}(L) \quad \text{for all} \quad A \in \mathfrak{A}(\mathbf{B}). \tag{13}$$

of course the $U_{\mathbf{B}}(L)$ are not uniquely determined by (13).

We may ask next whether the Lorentz transformations can be represented by unitary operators [again denoted by $U(L)$] in an irreducible representation space of the algebra $\mathfrak{A}$. This is the situation assumed in the usual analytic apparatus of quantum field theory. We may call it "Lorentz invariance of the representation" as distinguished from the Lorentz invariance of the algebraic theory which was postulated in the introduction. From the discussion A it follows that this is only possible if all the states φ belonging to the representation have the property[34]

$$||\mathbf{B}_{(\varphi_L)} - \mathbf{B}_{\varphi}|| \to 0 \tag{14}$$

when the region $\mathbf{B}$ is moved to infinity keeping its shape fixed. In other words, for such a representation all partial states in far away regions must be Lorentz invariant. The requirement that the Lorentz transformations shall be represented by unitary operators in the Hilbert space (Lorentz invariance of the representation) is thus a very powerful restriction eliminating most of the representations discussed under A).

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[32] "Far away" means the limit of an infinite spacelike separation.

[33] An automorphism of the type (12) is called an inner automorphism of the algebra. Our argument here shows that the Lorentz transformations are outer automorphisms.

[34] The symbol $\mathbf{B}_{\varphi}$ is used here to denote the partial state in $\mathbf{B}$ resulting from the restriction of φ.

In a Lorentz-invariant irreducible representation of $\mathfrak{A}$, the Lorentz operators $U(L)$ are, of course, obtainable as strong limits of the quasilocal operators since the strong closure of the collection of representatives of $\mathfrak{A}$ is the ring of all bounded operators on the Hilbert space. Consider a family of regions $\mathbf{B}_n$ such that $\mathbf{B}_{n+1} \supset \mathbf{B}_n$ and $\mathbf{B}_\infty = \bigcup \mathbf{B}_n$ is the whole of Minkowski space. The representatives of the corresponding family $U_{\mathbf{B}_n}(L)$ [see Eq. (13)] form a strongly (but not uniformly) converging sequence of operators due to the fact that every state in the representation space has the asymptotic property (14). The limit of the sequence is $U(L)$. In other representations in which the states have different asymptotic properties this sequence does not converge at all. This illustrates how global operations such as Lorentz transformations may be defined in suitable representation spaces as strong limits of quasilocal operations. The strong convergence of such a sequence of operators arises from the (common) asymptotic properties of all the states in the representation space. Strong convergence depends on the representation whereas uniform convergence does not.

C. Superselection Rules

In the usual formalism of Quantum field theory a superselection rule means that there are operators in the Hilbert space which commute with all observables. Typical examples of such "superselecting operators" are the total electric charge or the total baryon number. This customary representation of the algebra of observables is reducible. It can be decomposed into irreducible ones which we shall call "sectors." Each sector corresponds to a definite numerical value of the charge.[35]

We note first that the charge (as well as every other superselecting operator) is a global quantity. The distinction between the different sectors can therefore not be made by means of experiments in finite regions. A simple argument shows then that every sector is physically equivalent to every other sector. Let us demonstrate this for the two sectors corresponding to charge 3 and charge -1, respectively. We have to show that given an arbitrary state φ of charge 3 and an arbitrary finite set of elements $A_i \in \mathfrak{A}$ we can find a substitute state φ' with charge -1 so that the expectation values of the A_i in the two states differ by less than an arbitrarily prescribed tolerance ϵ. The way to construct φ' is physically evident. We change the physical situation described by φ adding 4 elementary particles of negative charge in a remote region of space. The effect of this added charge on the expectation values of the quasilocal quantities A_i tends to zero as the region is moved to infinity.

We conclude then from Fell's equivalence theorem that each single sector is a faithful representation of $\mathfrak{A}$.[36] A single sector contains already all relevant physical information. We note incidentally that we have here an example of a quasilocal algebra which has (at least) a denumerable infinity of unitarily inequivalent, Lorentz-invariant, faithful, irreducible representations (the various sectors).

In the standard treatment of field theory one considers the direct sum of all the sectors. If $\mathfrak{H}_n$ is the representation space of the sector with charge n then one uses the Hilbert space

$$\mathfrak{H} = \sum\nolimits^{\oplus} \mathfrak{H}_n. \tag{15}$$

Let us denote the representation of $\mathfrak{A}$ in $\mathfrak{H}$ by R, the range of this representation (i.e., the set of operators in $\mathfrak{H}$ representing the elements of $\mathfrak{A}$) by $R(\mathfrak{A})$. It is instructive to observe now the difference between weak and uniform closure. Since $\mathfrak{A}$ is a C^*-algebra $R(\mathfrak{A})$ is already uniformly closed. In the decomposition (15) the general element is of the form

$$R(A) = \begin{bmatrix} \ddots & & & \\ & R_{-1}(A) & & 0 \\ & & R_0(A) & \\ & 0 & & R_1(A) \\ & & & & \ddots \end{bmatrix} \tag{16}$$

where R_n is the irreducible representation of sector n. Since each R_n is faithful, these irreducible representations are rigidly coupled together. In other words, a single one of the suboperators $R_n(A)$ uniquely determines A and hence it fixes all the other $R_m(A)$. Thus in particular the projection operator P_n on the subspace $\mathfrak{H}_n$ does not belong to $R(\mathfrak{A})$ because the only element of $R(\mathfrak{A})$ which is zero in some sector is the zero operator on $\mathfrak{H}$. Let us consider now the weak (or strong) closure of $R(\mathfrak{A})$. This is the von Neumann ring generated by $R(\mathfrak{A})$ and can be alternatively obtained as the bicommutant $R(\mathfrak{A})''$. Since the representations R_n are irreducible and unitarily inequivalent Schur's lemma implies that the commutant $R(\mathfrak{A})'$ consists

[35] For the sake of simplicity we pretend that the electric charge is the only superselected quantity and thus use the word "charge" in lieu of "superselected quantities."

[36] The theorem tells us that one sector is equally as faithful as the collection of all sectors taken together.

of all operators of the form $\sum c_n P_n$ where the c_n are an arbitrary bounded sequence of complex numbers and P_n is the projector on $\mathfrak{H}_n$. Taking the commutant again we find that a general element of $R(\mathfrak{A})''$ is of the form

$$K = \begin{bmatrix} \ddots & & & & \\ & K_{-1} & & & \\ & & K_0 & 0 & \\ & 0 & & K_1 & \\ & & & & \ddots \end{bmatrix} \tag{17}$$

where the K_n are arbitrary bounded operators on the corresponding sectors (which can be chosen completely independent of each other). Thus the weak (or strong) closure of $R(\mathfrak{A})$ is the (uncoupled) product[37] of all the full matrix rings $\mathfrak{B}(\mathfrak{H}_n)$. It contains in particular all the projectors P_n, all the bounded functions of the charge as well as the Lorentz operators $U(L)$ ("global" quantities).

IV. COMPARISON WITH OPERATOR APPROACH TO QUANTUM FIELD THEORY

The postulates of a purely algebraic theory which have been stated so far [items (1) through (6) in the introduction, and (i) and (ii) in Sec. III] are not as powerful as those in other approaches to quantum field theory (Wightman's axioms or those of Ref. 2). In some respects this is good because a few irrelevant restrictions which are customarily imposed are eliminated. In other respects, however, the scheme as presented here is quite incomplete. It does not yet contain a stability condition and we have not formulated the counterparts of the finer structure properties which can be stated in the operator form of the theory. We shall point out now some of the features which are lacking in the present formulation.

The bridge between the algebraic approach and the customary analytic apparatus is the assumption that there exists a state Φ_0 over the algebra $\mathfrak{A}$ which is called the physical vacuum state and is supposed to have the following properties:

(α) Φ_0 is Lorentz-invariant, i.e., $(\Phi_0)_L = \Phi_0$.

(β) Φ_0 is a vector state of an irreducible, faithful representation of $\mathfrak{A}$ in a separable Hilbert space $\mathfrak{H}$.[38]

(γ) The Hamiltonian[39] is a positive-semidefinite operator in $\mathfrak{H}$.

It appears to us that the existence of a vacuum state with properties (α) and (β) is no *sine qua non* for a physically meaningful theory. In particular, in a theory describing among other things particles with zero rest mass one may have doubts as to whether the assumptions (α) and (β) are physically reasonable.[40] On the other hand, it is clear that some stability condition like (γ) is absolutely essential. The condition (γ) has also been one of the most useful tools in Wightman's approach.

Let us compare now the algebraic approach with that of Ref. 2 which is conceptually almost identical but uses von Neumann rings $\mathbf{R}(\mathbf{B})$ instead of abstract C^*-algebras $\mathfrak{A}(\mathbf{B})$. Given any specific irreducible, faithful representation of $\mathfrak{A}$ (say the representation R_α) we have immediately also a system of von Neumann rings which we denote by $\mathbf{R}_\alpha(\mathbf{B})$ in the representation space $\mathfrak{H}_\alpha$. The ring $\mathbf{R}_\alpha(\mathbf{B})$ is just the weak closure of the concrete C^*-algebra of operators $R_\alpha\{\mathfrak{A}(\mathbf{B})\}$ [the representatives of $\mathfrak{A}(\mathbf{B})$ in the representation R_α]. This ring system will satisfy the conditions of locality and causality; namely $\mathbf{R}_\alpha(\mathbf{B}_1)$ and $\mathbf{R}_\alpha(\mathbf{B}_2)$ commute if $\mathbf{B}_1$ and $\mathbf{B}_2$ are causally disjoint, and $\mathbf{R}_\alpha(\mathbf{B}_2) \subset \mathbf{R}_\alpha(\mathbf{B}_1)$ if $\mathbf{B}_2$ is the causal shadow of $\mathbf{B}_1$. This follows immediately from the corresponding relations for the algebras $\mathfrak{A}(\mathbf{B})$. However, within the set of von Neumann rings one has one important operation which has no direct counterpart in our family of C^*-algebras. This is the passage from a ring $\mathbf{R}$ to its commutant R'. This operation has been extensively used in Ref. 2 to formulate more detailed structure relations of the ring system which are very interesting because they open a way for a discussion of gauge groups and a distinction between theories with interaction from trivial theories in terms of local observables.[41] One typical example of such a relation is the "additivity"

$$\mathbf{R}(\mathbf{B}_1 \cup \mathbf{B}_2) = \{\mathbf{R}(\mathbf{B}_1), \mathbf{R}(\mathbf{B}_2)\}''. \tag{18}$$

Considering only the special case in which $\mathbf{B}_1$ and $\mathbf{B}_2$ are causally disjoint this relation may be assumed

[37] In the sense of Dixmier: *Les Algèbres d'operateurs dans l'espace Hilbertien* (Gauthier-Villars, Paris, 1952), Chap. I, Sec. 2.2.

[38] The representation is determined by Φ_0 via the GNS construction. See Appendix I, Ref. 59. It is irreducible if Φ_0 is pure. The separability would follow from the irreducibility if the algebra were separable in the norm topology.

[39] It follows from (α) that the representation obtained by the GNS construction from Φ_0 is Lorentz-invariant. Hence there exists a 1-parameter group of unitary operators $U(\tau)$ representing the time translations in $\mathfrak{H}$. The Hamiltonian is the infinitesimal generator of this group.

[40] This question was studied in Ref. 8 but the argument given there is inconclusive in some respects.

[41] For some conjectures in this direction see R. Haag, Ann. Physik **11**, 29 (1963) and Proceedings of the Conference on Analysis in Function Spaces, Massachusetts Institute of Technology, Cambridge, Massachusetts, 1963.

to hold for theories without superselection rules but to fail, for instance, in a theory with charged fields.

The question is therefore whether such relations as (18) can be given a meaning in the purely algebraic approach, in other words, whether they are independent of the particular representation α. Since the double commutant is the same as the weak closure this would be the case if all (faithful) representations of $\mathfrak{A}$ were *"locally quasi-equivalent."* The notion of "quasi-equivalence" of two representations was introduced by Mackey and is described in Appendix I. It is more restrictive than weak equivalence and less restrictive than unitary equivalence. By *"local"* quasi-equivalence of two representations we mean that for any (finite) region **B** the two representations of the subalgebra $\mathfrak{A}(\mathbf{B})$ are quasi-equivalent whereas the two representations of the full algebra $\mathfrak{A}$ need not be quasi-equivalent.

As pointed out in the Appendix, one may characterize quasi-equivalence also by the fact that the sets of states which appear as density matrices in the different representations are identical. Thus, the assumption of local quasi-equivalence of all representations means that each finite region **B** has a (universal) set of partial states which corresponds to the collection of density matrices in an arbitrary (faithful) representation of $\mathfrak{A}$ restricted to $\mathfrak{A}(\mathbf{B})$. In more intuitive language this assumption means two things. On the one hand, the results of measurements in a fixed finite region **B** shall be *uniformly* unaffected if one changes the state by "adding particles behind the moon," i.e.,

$$|\Phi(A) - \Phi'(A)| < \epsilon \, ||A|| \quad \text{for all} \quad A \in \mathfrak{A}(\mathbf{B}), \qquad (19)$$

if Φ' results from Φ by a unitary operation in a very distant region. Secondly, there shall be no other limiting procedure leading to the construction of inequivalent irreducible representations of $\mathfrak{A}$ besides the one involving large separation in position space and discussed in the last section. In particular one might wonder about the asymptotic limit for high energies. Is it possible to have states which differ in the asymptotic tail of their high-energy behavior (i.e., which give different expectation values for local operations involving "infinitely high" energy transfer)? The answer is probably no. Thus the assumption of local quasi-equivalence seems to us at the moment not unreasonable.

Another even stronger, assumption which is not contradicted by any of our present knowledge is that of *"local unitary equivalence"* of all irreducible, faithful representations of $\mathfrak{A}$. This would mean that if R and S are two such representations of $\mathfrak{A}$ then $R(\mathfrak{A}(\mathbf{B}))$ and $S(\mathfrak{A}(\mathbf{B}))$ are unitarily equivalent for every finite **B**, however, in such a way that no intertwining operator exists which is independent of the region **B**.

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APPENDIX I. C*-ALGEBRAS[42]

A complex (or real) *algebra* $\mathfrak{A}$ is a complex (or real) linear space such that to each ordered pair A, $B \in \mathfrak{A}$ there corresponds an element $A\,B \in \mathfrak{A}$, called their *product*, which is bilinear and associative (in general not commutative). In the special case where $AB = BA$ for all A, $B \in \mathfrak{A}$, $\mathfrak{A}$ is said to be *Abelian*. $\mathfrak{A}$ is a *-algebra[43] if to each $A \in \mathfrak{A}$ there corresponds a $A^* \in \mathfrak{A}$, called the *adjoint* of A, so that $A \rightarrow A^*$ is a conjugate-linear mapping with the properties $A^{**} = A$ and $(AB)^* = B^*A^*$ for any A, $B \in \mathfrak{A}$. $\mathfrak{A}$ is a *normed algebra* if to each $A \in \mathfrak{A}$ there corresponds a nonnegative number $||A||$, called the *norm* of A, in such a way that $||A|| > 0$ whenever $A \neq 0$ and, for any two A, $B \in \mathfrak{A}$ and any number λ, $||A + B|| \leq ||A|| + ||B||$, $||\lambda A|| = |\lambda| \, ||A||$ and $||AB|| \leq ||A|| \cdot ||B||$. Taking $||A - B||$ to be the distance of two elements A, B one defines on $\mathfrak{A}$ the topology of a metric vector space called its *uniform* or *norm topology*. If $\mathfrak{A}$ is both a *-algebra and a normed algebra we call it a **-normed algebra*[44] provided $||A^*|| = ||A||$ for all $A \in \mathfrak{A}$. A very important class of *-normed algebras, that of *Banach *-algebras*[45] is obtained by requiring completeness in the norm topology (i.e., convergence of all Cauchy sequences of elements of $\mathfrak{A}$ with respect to the norm to some element in $\mathfrak{A}$). One easily sees that any norm-closed *-algebra of bounded

[42] For general sources of information on C^*-algebras see M. A. Neumark, *Normierte Algebren* (VEB Deutscher Verlag der Wissenschaften, Berlin, 1959), Chap. V, Sec. 24; and C. E. Rickart, *General Theory of Banach Algebras* (D. Van Nostrand, Inc., New York, 1960), Chap. IV, Sec. 8. In general we use Rickart's terminology.

[43] Called *symmetrische Algebra* by Neumark.

[44] In Rickart's terminology. Neumark's is: *normierte symmetrische Algebra.*

[45] In Rickart's terminology. Neumark's is: *vollständige normierte symmetrische Algebra.*

operators on a Hilbert space is a Banach *-algebra (with respect to the usual addition, scalar multiplication, product, adjoint operation, and norm of bounded operators). However, the converse is not true and it has been shown by Gelfand and Neumark[46] that by requiring $||A^*A|| = ||A||^2$ for all $A \in \mathfrak{A}$ one singles out those particular Banach *-algebras which are isomorphic to (i.e., concretely realizable as) norm-closed *-algebras of bounded operators on some Hilbert space. These algebras are the interesting ones for quantum theory and we will call them *C^*-algebras*.[47] A typical example of a C^*-algebra is the algebra $\mathfrak{B}(\mathfrak{H})$ of all bounded operators on some Hilbert space $\mathfrak{H}$.

A linear mapping L of an algebra $\mathfrak{A}$ into an algebra $\mathfrak{A}_1$ which preserves products is called a *homomorphism*. If $\mathfrak{A}$ and $\mathfrak{A}_1$ are *-algebras and adjoints are mapped into adjoints, we speak of a **-homomorphism*. The *kernel* of the homomorphism L is the set of elements of $\mathfrak{A}$ which are mapped into the zero of $\mathfrak{A}_1$. A (*-)homomorphism is one-to-one if and only if its kernel reduces to zero, in which case it is called a **-isomorphism*. A linear subspace J of an algebra $\mathfrak{A}$ is a *left* (resp. *right*, resp. *two-sided*) *ideal* if $A \in \mathbf{J}$ and $B \in \mathfrak{A}$ imply $BA \in \mathbf{J}$ (resp. $AB \in \mathbf{J}$, resp. BA and $AB \in \mathbf{J}$). If $\mathfrak{A}$ is a *-algebra and $A \in \mathbf{J}$ implies $A^* \in \mathbf{J}$, we speak of a **-ideal*. Two-sided ideals (*-ideals) are in one-to-one correspondence with homomorphisms (*-homomorphisms), the first being the kernels of the latter. Given a two-sided ideal (*-ideal) $\mathbf{J}$ of $\mathfrak{A}$, the corresponding homomorphism (*-homomorphism) is obtained by assigning to each element $A \in \mathfrak{A}$ its class modulo the elements of $\mathbf{J}$. The algebra (*-algebra) of these equivalence classes is denoted $\mathfrak{A}/\mathbf{J}$ and called the *quotient of* $\mathfrak{A}$ *by* $\mathbf{J}$. A *representation* R of the algebra $\mathfrak{A}$ on a linear space $\mathfrak{H}$ is a homomorphism of $\mathfrak{A}$ into the algebra of linear operators on $\mathfrak{H}$: to each $A \in \mathfrak{A}$ R assigns a linear operator $R(A)$ on $\mathfrak{H}$ the correspondence $A \rightarrow R(A)$ respecting linear combinations and products. $R(A)$ is called the *representative* of A in R and the set of representatives of all $A \in \mathfrak{A}$ in R is called the *range* of R. R is *faithful* if it is one-to-one and *algebraically irreducible* if the only subspaces of $\mathfrak{H}$ invariant for all $R(A)$ are $\{0\}$ and $\mathfrak{H}$ itself. In the case where $\mathfrak{A}$ is a *-algebra, $\mathfrak{H}$ is a Hilbert space and R is a *-homomorphism into $\mathfrak{B}(\mathfrak{H})$ we speak of a **-representation* and call R (*topologically*) *irreducible*[48] if $\{0\}$ and $\mathfrak{H}$ are the only *closed* subspaces of $\mathfrak{H}$ invariant for all $R(A)$. Furthermore, we call R *continuous* in case $||R(A)|| \leq C\,||A||$ for all $A \in \mathfrak{A}$ and some positive constant C. One shows that every *-representation of a Banach *-algebra[49] is continuous. An ideal $\mathbf{J} \subset \mathfrak{A}$ is called *primitive* if it is the kernel of an algebraically irreducible representation. $\mathfrak{A}$ itself is *primitive* if $\{0\}$ is a primitive ideal, i.e., if $\mathfrak{A}$ admits a faithful algebraically irreducible representation. The *radical* of an algebra $\mathfrak{A}$ is the intersection of all its primitive ideals. In the case where $\mathfrak{A}$ is a *-algebra its **-radical* is the intersection of the kernels of all topologically irreducible *-representations. $\mathfrak{A}$ is *simple* if it contains only the trivial ideals $\{0\}$ and $\mathfrak{A}$ itself. It is *semisimple* (**-semisimple*[50]) if its radical (*-radical) reduces to zero. Simplicity obviously implies that all representations are faithful which implies primitivity in the case of a Banach *-algebra. Semisimplicity (*-semisimplicity) means only that the collection of all algebraically irreducible representations (topologically irreducible *-representations) is *faithful* in the sense that no two different elements of $\mathfrak{A}$ can have the same representative in all of them (we also say that the irreducible representations *separate* $\mathfrak{A}$).

The relation between separation properties of representations and the ideal structure of $\mathfrak{A}$ is greatly simplified in the case of C^*-algebras due to several important peculiarities. First, every *-isomorphism of a C^*-algebra into another C^*-algebra is norm-preserving. Second, every closed ideal $\mathbf{J}$ is a *-ideal and the corresponding quotient algebra $\mathfrak{A}/\mathbf{J}$ (equipped with its natural norm

$$||A + \mathbf{J}|| = \inf_{h \in \mathbf{J}} ||A + h||)$$

is itself a C^*-algebra.[51] Combining these facts, one finds that every *-homomorphism of a C^*-algebra has a uniformly closed range. In particular, the range of a *-representation of a C^*-algebra is itself a C^*-algebra, the representation being norm-pre-

[46] I. M. Gelfand and M. A. Neumark, Mat. Sb. **12**, 197 (1943).

[47] Here we depart from Rickart's terminology who calls *B^*-algebra* the abstract C^*-algebra and reserves the term *C^*-algebra* for a concrete norm-closed algebra of operators on a Hilbert space (we speak in this case of a *concrete C^*-algebra*). Our C^*-algebras (Rickart's B^*-algebras) are called by Neumark *vollreguläre vollständige Algebren*. Note that the condition $||A^*A|| = ||A||^2$ is evidently fulfilled in an operator algebra and that the distinction between abstract and concrete *C^*-algebras* is important because different concrete *C^*-algebras* can define the same abstract C^*-algebra.

[48] We always use the word irreducible to mean topologically irreducible.

[49] Whether or not $\mathfrak{A}$ contains an identity. See Rickart (Ref. 42), Theorem (4.1.20).

[50] Neumark uses *reduziert* for *-semisimple and *reduzierendes Ideal* for *-radical (in the case of Banach *-algebras).

[51] M. A. Neumark, Ref. 42, Chap. V, Sec. 24, Theorems 6 and 3.

serving if it is faithful. Further, a C^*-algebra is always semisimple and $*$-semisimple (its radical and $*$-radical both reducing to zero).[52] Another important result, due to Kadison,[53] is that every topologically irreducible $*$-representation of a C^*-algebra is algebraically irreducible. Finally, every algebraically irreducible representation of a C^*-algebra in a complex linear space is algebraically equivalent to a $*$-representation in a Hilbert space. Therefore every primitive ideal is the kernel of an irreducible $*$-representation. A primitive C^*-algebra can accordingly be defined as a C^*-algebra having at least one faithful irreducible $*$-representation.

Two representations ($*$-representations) R and S of $\mathfrak{A}$ on the respective spaces $\mathfrak{H}$ and $\mathfrak{K}$ are called *algebraically* (*unitarily*) *equivalent* if there exists a regular linear (unitary) operator U from $\mathfrak{H}$ onto $\mathfrak{K}$ such that

$$UR(A) = S(A)U \quad \text{for all} \quad A \in \mathfrak{A}. \tag{20}$$

Any linear (bounded linear) operator U from $\mathfrak{H}$ into $\mathfrak{K}$ satisfying (20) is called an *intertwining operator* for R and S. The set of such intertwining operators will be denoted by $\mathfrak{R}(R, S)$.[54] In the case of $*$-representations, one of the two following situations prevails: either $\mathfrak{R}(R, S) = \{0\}$ or there exist invariant closed subspaces $\mathfrak{H}_1$ and $\mathfrak{K}_1$ such that the restrictions of R and S on those subspaces (called subrepresentations of R and S) are unitarily equivalent. In particular, if R and S are irreducible and inequivalent, $\mathfrak{R}(R, S) = \{0\}$ and $\mathfrak{R}(R, R)$ consists of the multiples of unity (those two facts being generalizations of Schur's lemma). Note that $\mathfrak{R}(R, R)$ is a von Neumann ring, namely the commutant of $R(\mathfrak{A})$.

Two $*$-representations R and S of $\mathfrak{A}$ will be called *disjoint*[55] if $\mathfrak{R}(R, S) = \{0\}$, i.e., if R and S contain no subrepresentations which are unitarily equivalent. We consider now the decomposition of a representation into disjoint parts. Let R be a $*$-representation on $\mathfrak{H}$ and $\mathfrak{H}_1$ be a (closed) subspace of $\mathfrak{H}$ with projector E. $\mathfrak{H}_1$ is invariant in R if and only if $E \in \mathfrak{R}(R, R)$. In that case its orthogonal complement $\mathfrak{H}_1^\perp$ is also invariant and the restrictions of R on $\mathfrak{H}_1$ and $\mathfrak{H}_1^\perp$ (subrepresentations) are *disjoint* if and only if E is in the center of $\mathfrak{R}(R, R)$ (the center of an algebra being the set of its elements which commute with all others). A $*$-representation R of $\mathfrak{A}$ is called *primary*[55] if the center of $\mathfrak{R}(R, R)$ contains only the multiples of unity (so if no two subrepresentations of R are disjoint). We have seen that the range $R(\mathfrak{A})$ of R is closed in the norm-topology of operators on $\mathfrak{H}$, but it is in general not closed in the weak topology of operators on $\mathfrak{H}$ (the weak closure of $R(\mathfrak{A})$ is the von Neumann ring $R(\mathfrak{A})''$ which contains in general many more operators than $R(\mathfrak{A})$ itself). R is primary if $R(\mathfrak{A})''$ is a *factor* in the sense of von Neumann.

Two representations R and S of $\mathfrak{A}$ are called *quasi-equivalent*[56] if they have the same kernel (i.e., if their ranges are $*$-isomorphic) and if this $*$-isomorphism extends to the weak closures of the ranges in the respective weak topologies of operators on the representation spaces. The $*$-representations R and S are quasi-equivalent if and only if no subrepresentation of the one is disjoint from the other. If R and S are primary then they are either quasi-equivalent or disjoint.

A role of primary importance in the study of a Banach $*$-algebra is played by its *positive forms*. A linear form Φ on $\mathfrak{A}$ is called *continuous* if $|\Phi(A)| \leq C\,||A||$ for all $A \in \mathfrak{A}$ and some positive constant C. The smallest such constant is denoted by $||\Phi||$ and called the *norm* of Φ. Under this norm the set of all continuous forms on $\mathfrak{A}$ is a Banach space called the *dual space of* $\mathfrak{A}$ and denoted by $\mathfrak{A}^*$. (The Banach space topology of $\mathfrak{A}^*$ is called its *uniform* or *norm topology*.) Another topology of interest on $\mathfrak{A}^*$ is its *weak topology* (with respect to $\mathfrak{A}$) characterized by the pseudo-norms $N_A(\Phi) = |\Phi(A)|$ where A runs through $\mathfrak{A}$ (or by the complete set of neighborhoods of zero $V_{\{A_i,\epsilon\}}$, $A_1, A_2, \cdots, A_n \in \mathfrak{A}$, $\epsilon > 0$, where $V_{\{A_i,\epsilon\}}$ consists of all $\Phi \in \mathfrak{A}^*$ such that $|\Phi(A_i)| < \epsilon$, $i = 1, 2, \cdots, n$). A linear form Φ is positive if $\Phi(A^*A) \geq 0$ for all $A \in \mathfrak{A}$. If $\mathfrak{A}$ has a unit I the continuity of Φ follows from the positivity and one has $||\Phi|| = \Phi(I)$. Positive forms are also ipso facto continuous for C^*-algebras with or without unit.[57] The set of all positive continuous forms on $\mathfrak{A}$ (or *states* on $\mathfrak{A}$) is called the *positive cone* of $\mathfrak{A}^*$ and denoted by $\mathfrak{A}^{*(+)}$. We will denote by $\mathbf{\Sigma}$ and $\dot{\mathbf{\Sigma}}$ the subsets consisting of all $\Phi \in \mathfrak{A}^*$ such that $||\Phi|| \leq 1$ and $||\Phi|| = 1$, respectively ($\mathbf{\Sigma}$ is the *unit ball* of $\mathfrak{A}^*$). By Tychonov's theorem $\mathfrak{A}^{*(+)} \cap \mathbf{\Sigma}$ (and $\mathfrak{A}^{*(+)} \cap \dot{\mathbf{\Sigma}}$ if $\mathfrak{A}$ has a unit) are compact subsets of $\mathfrak{A}^*$ in its weak topology. As a compact convex set $\mathfrak{A}^{*(+)} \cap \dot{\mathbf{\Sigma}}$ has extremal elements and is equal

[52] M. A. Neumark, Ref. 42, Chap. V, Sec. 24, Theorem 4.
[53] R. V. Kadison, Proc. Natl. Acad. Sci. U. S. **43**, 273 (1957).
[54] See G. W. Mackey, "The Theory of Group Representations," University of Chicago, mimeographed lecture notes.
[55] See Mackey, Ref. 54.
[56] For the notion of quasi-equivalence, see Mackey, Ref. 54, Chap. I.
[57] See C. E. Rickart, Ref. 42, Theorems (4.5.14), (4.5.11), and (4.8.14).

to the weak closure of their convex hull (theorem of Krein-Milman). The state $\Phi \in \mathfrak{A}^{*(+)} \cap \dot{\Sigma}$ is called *extremal* or *pure* if it cannot be written as $\lambda_1\Phi_1 + \lambda_2\Phi_2$ with $\Phi_1, \Phi_2 \in \mathfrak{A}^{*(+)} \cap \dot{\Sigma}$, $\Phi_1 \neq \Phi_2$, $0 < \lambda_1 < 1$, $\lambda_1 + \lambda_2 = 1$).

The importance of positive linear forms for Banach *-algebras lies in their connection with *-representations. Each *-representation of a *-algebra is (by transfinite induction) the direct sum of *-representations all of which (except the null representation) are cyclic.[58] Now for a Banach *-algebra with approximate unit, in particular for a C^*-algebra, giving a positive (*ipso facto* continuous) linear form Φ amounts to the same thing as specifying a cyclic representation R and a cyclic vector ξ. This representation is irreducible if and only if Φ is pure. Given R and ξ we have $\Phi(A) = (\xi\,|R(A)|\,\xi)$. Conversely, given Φ, its *null space* $\mathfrak{N}$ (i.e., the set of elements $A \in \mathfrak{A}$ for which $\Phi(A^*A) = 0$ or, equivalently, for which $\Phi(A^*B) = 0$ for all $B \in \mathfrak{A}$) is a left ideal in $\mathfrak{A}$ and one recovers consistently the vectors $R(A)\xi$, their scalar products $(R(A)\xi \mid R(B)\xi)$, and the operator $R(C)$ acting on $R(A)\xi$ by the following identification:

vector $R(A)\xi \leftrightarrow$ class (modulo $\mathfrak{N}$) of the algebraic elements $A + \mathfrak{N}$,
scalar product $(R(B)\xi \mid R(A)\xi) = \Phi(B^*A)$,
action of operator $R(C)R(A)\xi \leftrightarrow CA + \mathfrak{N}$.

$\mathfrak{H}$ is then constructed by completion and the operator $R(C)$ in the complete $\mathfrak{H}$ by continuous extension. The cyclic vector ξ corresponds to $I + \mathfrak{N}$ if I exists and is otherwise obtained with the help of an approximate unity.[59]

It is useful to characterize the relations between different *-representations of a C^*-algebra in terms of certain subsets of $\mathfrak{A}^*$ determined by their representation spaces. Let R be a *-representation of $\mathfrak{A}$ on the Hilbert space $\mathfrak{H}_R$. Given $\Psi \in \mathfrak{H}_R$ and $A \in \mathfrak{A}$ we denote by $\omega_\Psi(A)$ the expectation value of $R(A)$ in the vector Ψ:

$$\omega_\Psi(A) = (\Psi|\,R(A)\,|\Psi) = \mathrm{Tr}\,R(A)\,|\Psi)(\Psi|.$$

We have thus defined a positive form ω_Ψ on $\mathfrak{A}$ which we call the *vector state determined by* $\Psi \in \mathfrak{H}_R$. When Ψ runs through $\mathfrak{H}_R$, ω_Ψ runs through a subset $\omega(R)$ of $\mathfrak{A}^{*(+)}$ which can be shown to be uniformly closed in $\mathfrak{A}^*$. For cyclic representations $\omega(R)$ determines R up to unitary equivalence; for two cyclic representations R and R' of $\mathfrak{A}$ to be unitarily equivalent, it is necessary and sufficient that $\omega(R) = \omega(R')$.[60] We now pass from $\omega(R)$ to its convex hull conv $\{\omega(R)\}$ (i.e., we consider all finite linear combinations of its elements with positive coefficients). If we close this convex hull respectively in the uniform and in the weak topology of $\mathfrak{A}^*$, we get two sets of states $\overline{\overline{\mathrm{conv}}}\,\{\omega(R)\}$ and $\overline{\mathrm{conv}}\,\{\omega(R)\}$ which respectively determine R up to quasi-equivalence and weak equivalence.[61] The elements of the uniform closure $\overline{\overline{\mathrm{conv}}}\,\{\omega(R)\}$ can be characterized as the set of states Φ of the form

$$\Phi(A) = \mathrm{Tr}\,(\Phi_{\mathrm{op}}\cdot A) \tag{21}$$

where Φ_{op} is any positive linear operator on $\mathfrak{H}_R$ with finite trace (the norm $||\Phi||$ being equal to the trace of Φ_{op}. These states will be referred to as the *density matrices* in the representation R. The elements of conv $\{\omega(R)\}$ are correspondingly the *density matrices of finite rank* in R. The linear spans of $\overline{\overline{\mathrm{conv}}}\,\{\omega(R)\}$ and conv $\{\omega(R)\}$ are obtained by taking Φ_{op} in (21) to be respectively the operators of the trace class and of finite rank on $\mathfrak{H}_R$. We will denote them accordingly by $\mathfrak{T}(\mathfrak{A}, R)$ and $\mathfrak{F}(\mathfrak{A}, R)$. One has $\mathfrak{T}^+(\mathfrak{A}, R) = \overline{\overline{\mathrm{conv}}}\,\{\omega(R)\}$ and $\mathfrak{F}^+(\mathfrak{A}, R) = \mathrm{conv}\,\{\omega(R)\}$ where $^+$ indicates the restriction to positive elements. Let $t^S(R)$, $t^s(R)$, $t^W(R)$, $t^w(R)$ denote the topologies respectively defined on $\mathfrak{A}$ by the strongest, the strong, the σ-weak and the weak topologies of operators on $\mathfrak{H}_R$ (those topologies are not separating if R is not faithful). $\mathfrak{T}(\mathfrak{A}, R)$, resp. $\mathfrak{F}(\mathfrak{A}, R)$ $[\mathfrak{T}^+(\mathfrak{A}, R)$, resp. $\mathfrak{F}^+(\mathfrak{A}, R)]$ can be characterized as the set of linear forms on $\mathfrak{A}$ (of positive linear forms on $\mathfrak{A}$) continuous with respect to either $t^S(R)$ or $t^s(R)$, resp. either $t^W(R)$ or $t^w(R)$. So quasi-equivalence of R and R' means that $t^S(R) = t^S(R')$, or equivalently $t^s(R) = t^s(R')$.

Let S and T be two *-representations of the C^*-algebra $\mathfrak{A}$, and let Ker (S) and Ker (T) be their kernels. Fell's equivalence theorem states that Ker $(S) \supseteq$ Ker (T) is equivalent to $\omega(S) \subseteq \overline{\mathrm{conv}}\,\{\omega(T)\}$, or alternatively to $\omega(S) \cap \Sigma \subseteq \overline{\mathrm{conv}}\,\{\omega(T) \cap \Sigma\}$ or again to $\omega(S) \cap \dot{\Sigma} \subseteq \overline{\mathrm{conv}}\,\{\omega(T) \cap \Sigma\}$. If this is the case Fell calls S *weakly contained* in T. If S and T are weakly con-

[58] A representation R in the space $\mathfrak{H}$ is called *cyclic* with *cyclic vector* $\xi \in \mathfrak{H}$ if the set of vectors $R(\mathfrak{A})\,\xi$ is dense in $\mathfrak{H}$.

[59] This construction is due to I. M. Gelfand and M. A. Neumark, Isvertija Ser. Mat. **12**, 445 (1948); and I. E. Segal, Bull. Am. Math. Soc. **53**, 73 (1947). We call it the GNS construction. See M. A. Neumark, Ref. 42, Chap. IV, Sec. 17.3 or, for the case of an algebra without unit, C. E. Rickart, Ref. 42, Chap. IV, Sec. 5.

[60] These results can be inferred from R. V. Kadison, Trans. Am. Math. Soc. **103**, 304 (1962). $\omega(R) \subseteq \omega(R')$ means that R is unitarily equivalent to some subrepresentation of R'.

[61] For this characterization of quasi-equivalence, see Z. Takeda, Tôhoku Mat. J. **6**, 212 (1954).

tained in each other they are *weakly equivalent* (for us *physically equivalent*). If S is cyclic with cyclic vector Ψ it is sufficient for S to be weakly contained in T that $\omega_\Psi \in \overline{\text{conv}}\,\{\omega(T)\}$. If S and T are irreducible and S is weakly contained in T then $\omega(S)$ is already contained in the weak closure of $\omega(T)$.

APPENDIX II. NONSIMPLICITY OF THE FERMION CURRENT ALGEBRA[62]

A. Finite-Dimensional Case

Let $\mathfrak{E}$ be an n-dimensional metric vector space over the complex numbers. Corresponding to each vector x from $\mathfrak{E}$ we consider two algebraic elements $a^*(x)$ and $a(x)$ (adjoints of each other). They shall satisfy the commutation relations of creation and destruction operators in Fermi statistics,

$$a^*(x)a(y) + a(y)a^*(x) = (x, y),$$
$$a^*(x)a^*(y) + a^*(y)a^*(x) = 0, \tag{22}$$

and the "creators" $a^*(x)$ shall depend linearly on x. The $*$-algebra generated by the a^* and a will be denoted by $\mathfrak{C}$.[63] A representation of $\mathfrak{C}$ is obtained (the standard representation in physical applications) by stipulating that the representation space $\mathfrak{G}(\mathfrak{E})$ shall contain a vector Φ_0 satisfying

$$a(x)\Phi_0 = 0 \quad \text{for all} \quad x \in \mathfrak{E} \tag{23}$$

and that the other vectors of $\mathfrak{G}(\mathfrak{E})$ are obtained from Φ_0 by application of polynomials of the a^*. From the commutation relations one infers immediately that the space $\mathfrak{G}$ has 2^n dimensions, that $\mathfrak{C}$ has 4^n linearly independent elements and is isomorphic to the full matrix ring over $\mathfrak{G}$. Thus $\mathfrak{C}$ is simple.

If $\Pi \in \mathfrak{C}$ is a product of p creators and q annihilators (in any order) we define the *grade of* Π to be the difference $p - q$. $\mathfrak{C}$ thus becomes a graded algebra whose zero-grade part will be called $\mathfrak{C}_0$. $\mathfrak{C}_0$ is represented in $\mathfrak{G}$ by operators which leave the homogeneous subspaces $\mathfrak{G}_p$ invariant. ($\mathfrak{G}_p$ is that subspace which is generated from Φ_0 by homogeneous polynomials of the a^* of order p; p runs from 0 to n.) Calling $R_p(\mathfrak{C}_0)$ the restriction of the representation of $\mathfrak{C}_0$ to $\mathfrak{G}_p$ one sees by counting dimensions that the R_p are a separating family of irreducible inequivalent representations of $\mathfrak{C}_0$. Therefore $\mathfrak{C}_0$ is a semisimple (but not simple) finite dimensional algebra. Note that $\mathfrak{C}_0$ can be regarded as the *"algebra of currents"* where the *"current"* $j(K)$ corresponding to the linear operator K on $\mathfrak{E}$ is defined by

$$j(K) = \sum_{i,k} \langle x_i \,|K|\, x_k\rangle a^*(x_i)a(x_k) \tag{24}$$

(x_i being a complete orthonormal basis of $\mathfrak{E}$).

B. Infinite-Dimensional Case

Instead of the finite-dimensional $\mathfrak{E}$ we take now an infinite-dimensional Hilbert space $\mathfrak{H}$. For any finite dimensional subspace $\mathfrak{E} \subset \mathfrak{H}$ we can consider the algebra $\mathfrak{C}(\mathfrak{E})$ has previously defined and one sees easily that for $\mathfrak{E}_2 \supset \mathfrak{E}_1$ the algebra $\mathfrak{C}(\mathfrak{E}_1)$ is is canonically embedded in $\mathfrak{C}(\mathfrak{E}_2)$ (as a normed $*$-algebra). Thus we can define $\mathfrak{C}(\mathfrak{H})$ as the completion of the union of all the $\mathfrak{C}(\mathfrak{E})$ for the finite-dimensional subspaces $\mathfrak{E} \subset \mathfrak{H}$. The fact that each $\mathfrak{C}(\mathfrak{E})$ is simple implies that all $*$-representations of $\mathfrak{C}(\mathfrak{H})$ are faithful and isometric, i.e., that also $\mathfrak{C}(\mathfrak{H})$ is simple.[64]

In the case of a free Dirac field $\mathfrak{H}$ is the direct sum of two spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ which correspond, respectively, to the states of a single electron and to those of a single positron. We are interested now in the zero grade part of $\mathfrak{C}(\mathfrak{H})$. This algebra $\mathfrak{C}_0$ may be regarded as the algebra of currents in the theory of a free Dirac field. We consider the two familiar irreducible representations of $\mathfrak{C}$:

(1) the old-fashioned one which results if we assume the existence of a state Φ_0 satisfying

$$a(x)\Phi_0 = 0 \quad \text{for all} \quad x \in \mathfrak{H}; \tag{25}$$

(2) the "charge symmetric" one in which one assumes a state Ψ_0 satisfying

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[62] We are indebted to Professor H. Araki for pointing out to us the main facts described in this appendix.

[63] $\mathfrak{E}$ is the Clifford algebra over the space $\mathfrak{E} \oplus \bar{\mathfrak{E}}$ with respect to the bilinear scalar product

$$g(x \oplus \bar{y}, x' \oplus \bar{y}') = \tfrac{1}{2}\,\{(x, y') + (y, x)\} \quad x \in \mathfrak{E}, y \in \mathfrak{E}.$$

Here $\bar{\mathfrak{E}}$ is the "complex conjugate" of $\mathfrak{E}$; i.e., it is isomorphic to $\mathfrak{E}$ as an additive group:

$$x \in \mathfrak{E} \leftrightarrow \bar{x} \in \bar{\mathfrak{E}},$$

but its scalar multiplication reverses the sign of i:

$$\overline{((\alpha + i\beta)x)} = (\alpha - i\beta)\bar{x}.$$

Since (x, y) is Hermitian symmetric the form g is bilinear. $\mathfrak{C}$ can be defined as the quotient of the tensor algebra over $\mathfrak{E} \oplus \bar{\mathfrak{E}}$ by an ideal $\mathfrak{T}$ which is generated by the tensors $\xi \times \xi - g(\xi, \xi)$ with $\xi \in \mathfrak{E} \oplus \mathfrak{T}$. One has $a^*(x) = x \oplus 0 \bmod \mathfrak{T}$ and $a(y) = 0 \oplus \bar{y} \bmod \mathfrak{T}$. By extension of the adjoint operation

$$x \oplus \bar{y} \leftrightarrow y \oplus \bar{x}$$

one defines on $\mathfrak{C}$ the structure of a $*$-algebra. Equation (23) defines a faithful realization of $\mathfrak{C}$ by the linear operators on the space $\mathfrak{H}(\mathfrak{E})$ which latter coincides with the Grassmann algebra over $\mathfrak{E}$.

[64] Since it is known to have many inequivalent irreducible representations it is an NGCR-algebra.

$$a(x)\Psi_0 = 0 \quad \text{for all} \quad x \in \mathfrak{H}_1,$$
$$a^*(x)\Psi = 0 \quad \text{for all} \quad x \in \mathfrak{H}_2. \tag{26}$$

Both representations, restricted to $\mathfrak{C}_0$, split up into irreducible parts corresponding to the different values of the charge. Now in Case 1 none of those subrepresentations of $\mathfrak{C}_0$ is faithful. In the subspace corresponding to charge n all operators having more than n annihilators on the right have zero representatives. Thus $\mathfrak{C}_0$ has nontrivial ideals and is accordingly not simple. On the other hand, in the charge-symmetric representation of $\mathfrak{C}$ (Case 2) all the subrepresentations of $\mathfrak{C}_0$ corresponding to a fixed value of the charge are faithful. This is an immediate consequence of the semisimplicity of the $\mathfrak{C}(\mathfrak{E})$ for finite-dimensional $\mathfrak{E}$.