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Strict Localization in Quantum Field Theory*

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A definition of strict localization of states in quantum field theory is presented. This definition is based on considering products of field operators as the primary measurable quantities of the theory. An example of a localized state is given, showing that such a state arises when a free field interacts with an external current that is limited to a bounded region of space-time. It is shown by means of a graphical technique that a state having a finite number of particles cannot satisfy the definition of localization. A simple representation of localized states is investigated, and arguments are given to support its generality and uniqueness.

1. INTRODUCTION

THE research in quantum field theory of the past ten years has centered chiefly about the analytic properties of various quantities appearing in the theory.[1] These properties are derived from very general characteristics of the fields stated in the form of postulates which it is believed that any complete theory must satisfy. It is not altogether clear, however, that these postulates, as formulated for example by Wightman,[2] form a consistent system, or that they form a minimal basis from which the analytic properties follow. It is, therefore, of great interest at the present time to carry out investigations having for their ultimate aim the clarification of these postulates.

The earliest derivations of analytic properties for relativistic field theories were based on the finite propagation velocity of wave disturbances, and took into consideration the scattering of initially separated localized wave packets.[3] These packets propagate towards one another with finite velocity and interact, giving rise to scattered waves. The condition that no scattered waves appear before the initial wave packets have had time to collide is then sufficient to give analytic properties for the scattering amplitude as a function of the wave number k.

In later derivations of analytic properties, carried out in the formalism of quantum field theory, the principle of finite propagation velocity was replaced by the condition of local commutativity of the field operators at space-like separations,[4] together with certain other postulates such as the asymptotic condition. This latter condition is an expression of the circumstance that particles involved in a scattering process behave as separated and noninteracting at times in the distant past and future before and after the scattering has taken place. It thus replaces the wave-packet description of the scattering process used in earlier derivations. The recent work of Haag[5] has gone far in clarifying the status of the asymptotic condition, although it is still not clear in precisely what form this condition is satisfied.

It is clear that the question of localization of states is fundamental to the above considerations. To gain an

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[1] For a review of this work, see for example, the account of the Verenna course on mathematical problems of the quantum theory of fields, published in Nuovo cimento **14**, Suppl. (1959).

[2] A. S. Wightman, Phys. Rev. **101**, 860 (1956), and Ecole Normale Lecture Notes (1957).

[3] J. S. Toll, thesis, Princeton University (1952), and Phys. Rev. **104**, 1760 (1957); N. G. van Kampen, *ibid.* **89**, 1073 (1953); **91**, 1267 (1953).

[4] M. Gell-Mann, M. L. Goldberger, and W. E. Thirring, Phys. Rev. **91**, 1612 (1954).

[5] R. Haag, *Les Problèmes Mathématique de la Théorie Quantique des Champs* (Lille, 1957); Phys. Rev. **112**, 669 (1958), and the article cited in reference 1. See also D. Kastler, Compt. rend. acad. sci. Paris **245**, 2021 (1957).

idea of what localization means, one must have an idea of the significance of the measuring process. Newton and Wigner took up the question of localization from the point of view of position measurements of a particle.[6] They set up postulates from which the position eigenstates of a particle could be determined. Their postulates were based on the quantum theoretical description of a measuring process and upon relativistic invariance of the wave function describing the particle. In order that the position measurement have meaning, eigenstates corresponding to different spatial positions at a fixed time must be orthogonal. This condition is sufficient to determine the position eigenstates completely. However, the definition of localization implied by these position eigenstates is found not to be preserved in time, i.e., a particle localized at a point at one time will be spread over all space, even outside the light cone of the initial point, at later times. Furthermore, a particle localized in one Lorentz frame is not necessarily localized in another. These rather unsatisfactory features of the results of Newton and Wigner make it difficult to see how a suitable concept of localization of particles may be defined in relativistic field theory.

Haag[5] has discussed the asymptotic condition in field theory by means of a definition of localization in which two states are localized if they become orthogonal as the space-like separation of their respective regions of localization becomes infinite. This definition is less restrictive than the one adopted in this paper, which is based on strict localization of a state determined by measurement.

A basic treatment of measurements in field theory was given by Bohr and Rosenfeld,[7] who showed that averages of the field variables over space-time regions may be taken as the basic measurable quantities of the theory. This view seems appropriate also to the recent postulational developments of field theory, where vacuum expectation values of certain combinations of field variables are treated as fundamental.

The definition of localization given in Sec. 2 is based upon taking products of field operators, instead of particle observables, as the basic measurable quantities. The remainder of the paper is devoted to an analysis of the definition and a discussion of some of the properties of the states satisfying the definition.

2. DEFINITION OF LOCALIZATION

In this section, we present and discuss the definition of localization that forms the basis of this paper. We take the point of view that the basic measurable quantities of the theory are products of field variables $A(x)$. By this we mean that any observable quantity Q shall be expressible in the following form as sums of integrals of such products:

$$Q=q_0+\int dx_1 q_1(x_1)A(x_1)+\int\int dx_1 dx_2 q_2(x_1,x_2) \times A(x_1)A(x_2)+\cdots, \quad (1)$$

q_i being c-number functions of their space-time arguments.[8] The expectation value of Q in any state $|\psi\rangle$ can then be written in terms of the quantities

$$\langle\Psi|\prod_{i=1}^{n} A(x_i)|\Psi\rangle$$

in a form similar to (1). We wish to formulate the condition that $|\Psi\rangle$ represent a state of the field which is strictly localized in a region $\mathfrak{U}$ of space-time. If the phenomena described by and the field quantity $A(x)$ are confined to $\mathfrak{U}$, it should not be possible to detect the presence of any field disturbance by making measurements at points outside of $\mathfrak{U}$. In other words, such measurements should lead to the same results whether the state of the system is $|\Psi\rangle$ or the vacuum state $|0\rangle$. Taking into consideration the nature of the measurable quantities (1), we arrive at the following definition:

Definition of localization. A state $|\Psi\rangle$ of a field $A(x)$ is localized in the region $\mathfrak{U}$ if

$$\langle\Psi|\prod_{i=1}^{n} A(x_i)|\Psi\rangle=\langle 0|\prod_{i=1}^{n} A(x_1)|0\rangle, \quad n=1, 2, \cdots, \quad (2)$$

for any product of field operators taken at points x_i all of which lie outside of $\mathfrak{U}$.

If there is more than one independent field in the theory, the definition may be extended to include all possible products of any combination of the fields. The definition is meant to apply to fields satisfying the Bose-Einstein statistics. For Fermi-Dirac fields, the measurable quantities are bilinear expressions[9] in the field operators, such as charge and current densities, and the definition must be modified accordingly. We have limited our investigation to the case of a single field satisfying Bose-Einstein statistics. This case contains the essential elements of the problem, and an extension to other cases should not present any difficulties.

For the most part, the analysis that follows is carried out for the case of free fields. The definition, however, is not so restricted, and it is pointed out explicitly wherever our arguments clearly have more general validity. We do not consider the limitations to free fields to be a very serious one, since we are interested in localized states primarily as initial and final states in scattering processes. We expect that in such cases the particles involved in the process are far separated and are therefore describable by free fields.

[6] R. G. Newton and E. P. Wigner, Revs. Modern Phys. **21**, 400 (1949).

[7] N. Bohr and L. Rosenfeld, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **12**, No. 8 (1933), and Phys. Rev. **78**, 794 (1950).

[8] Four-dimensional integrals will be denoted by the symbol dx, three-dimensional integrals by d^3x.

[9] N. Bohr and L. Rosenfeld, Phys. Rev. **78**, 794 (1950).

It is possible to make a statement about the nature of the region $\mathfrak{U}$ in which a state may be localized without entering into the details of the structure of the state itself. We note that the quantity

$$\Psi(x_1,\cdots,x_n)=\langle\Psi|\prod_{i=1}^{n}A(x_i)|\Psi\rangle-\langle 0|\prod_{i=1}^{n}A(x_i)|0\rangle$$

is a solution of the free field equation in each of its arguments independently. It follows from this that if $f(x)$ is an arbitrary solution of the Klein-Gordon equation vanishing sufficiently rapidly for large spatial separations at a given time, then the expression

$$\int_{x_0=t}d^3x_1\left(f(x_1)\frac{\partial}{\partial x_{10}}\Psi(x_1,\cdots,x_n)-\frac{\partial f(x_1)}{\partial x_{10}}\Psi(x_1,\cdots,x_n)\right)$$

is independent of t for fixed x_2, $\cdots$, x_n. This makes it clear that if $\Psi(x_1,\cdots,x_n)$ is not identically zero, it must be nonvanishing in some region of space at any given time. Since (2) requires $\Psi(x_1,\cdots,x_n)$ to vanish outside $\mathfrak{U}$, this means that $\mathfrak{U}$ cannot be bounded in the time-like direction.

The actual form of the region of localization may be also inferred from similar considerations. The solution $\Psi(x)=\langle\Psi|A(x)|\Psi\rangle-\langle 0|A(x)|0\rangle$ of the Klein-Gordon equation may be expressed in terms of its initial values on a surface $x_0=t$ as follows:

$$\Psi(x)=-\int_{x_0=t}d^3x\left(\Delta(x-x')\frac{\partial\Psi(x')}{\partial x_0'}-\frac{\partial\Delta(x-x')}{\partial x_0'}\Psi(x')\right)$$

by means of the singular function $\Delta(x-x')$.[10] If at a time t, $\Psi(x)$ and $\partial\Psi(x)/\partial x_0$ are confined to a region $\mathfrak{R}$ of space, then $\Psi(x)$ for arbitrary times will be confined to the region $V_+(\mathfrak{R})$ consisting of $\mathfrak{R}$ together with the interiors of all the forward and backward light cones with vertices in $\mathfrak{R}$. This is clear from the property that the Δ function vanishes for space-like argument. Note that $\mathfrak{R}$ may be taken as any bounded space-time region instead of a space-like surface without changing the manifold of possible regions of localization.

3. FIELD INTERACTING WITH CLASSICAL CURRENT DISTRIBUTION

To provide an example of a localized state, we consider the interaction of a field $\phi(x)$ with a classical current distribution $j(x)$. The equation of motion for $\phi(x)$ is[11]

$$(\Box-m^2)\phi(x)=-j(x).$$

We may express $\phi(x)$ as follows in terms of free fields $\phi_{\text{in}}(x)$ and $\phi_{\text{out}}(x)$

$$\phi(x)=\phi_{\text{in}}(x)+\int\Delta_R(x-x')j(x')dx'$$

$$=\phi_{\text{out}}(x)+\int\Delta_A(x-x')j(x')dx',$$

$$\phi_{\text{out}}=\phi_{\text{in}}(x)-\int\Delta(x-x')j(x')dx'.$$

We may now introduce two complete sets of states, obtained from the in and out fields, respectively. Each of these sets may be labeled by the eigenvalues of the number operators and the momentum operators. Because of the action of the current $j(x)$, the particle configuration will change with time so that at large times there will be a different configuration, represented as some linear combination of the outgoing states. For example,[11] if the state of the system is $|0_{\text{in}}\rangle$, i.e., no particles initially, then an emission of quanta will take place such that the probability of finding n outgoing quanta is the state of momentum $\mathbf{k}$ is given by the Poisson law

$$w_n(\mathbf{k})=(n!)^{-1}(\langle n(\mathbf{k})\rangle)^n\exp(-\langle n(\mathbf{k})\rangle),$$

where $\langle n(\mathbf{k})\rangle$, the average number of particles of momentum $\mathbf{k}$, is proportional to the quantity $|j(\mathbf{k},\omega)|^2$, where $\omega=+(\mathbf{k}^2+m^2)^{\frac{1}{2}}$, and $j(\mathbf{k},\omega)$ is the Fourier transform of $j(x)$. Note that the outgoing configuration depends only on those components of the Fourier transform which satisfy $\mathbf{k}^2-\omega^2=-m^2$, and not on components off the mass shell. We shall refer to this fact in Sec. 5.

We now make the following identification:

$$A(x)=\phi_{\text{out}}(x),$$
$$|0\rangle=|0_{\text{out}}\rangle,$$
$$|\Psi\rangle=|0_{\text{in}}\rangle.$$

We will now show that, if $j(x)=0$ outside of the space-time region $\mathfrak{R}$, then $|\Psi\rangle$ is a localized state of the field $A(x)$ in $V_+(\mathfrak{R})$ by definition (2). We evaluate the quantity

$$\langle\Psi|\prod_{i=1}^{n}A(x_i)|\Psi\rangle$$

$$=\langle 0_{\text{in}}|\prod_{i=1}^{n}\phi_{\text{out}}(x_i)|0_{\text{in}}\rangle$$

$$=\langle 0_{\text{in}}|\prod_{i=1}^{n}\left(\phi_{\text{in}}(x_i)-\int dx_i'\Delta(x_i-x_i')j(x_i')\right)|0_{\text{in}}\rangle$$

$$=\langle 0_{\text{in}}|\prod_{i=1}^{n}\phi_{\text{in}}(x_i)|0_{\text{in}}\rangle$$

$$-\sum_{j=1}^{n}\int dx_j'\Delta(x_j-x_j')j(x_j')$$

$$\times\langle 0_{\text{in}}|\prod_{i\neq j=1}^{n}\phi_{\text{in}}(x_i)|0_{\text{in}}\rangle+\cdots. \quad (3)$$

[10] See, for example, G. Källén's article in *Handbuch der Physik* (Springer-Verlag, Berlin, Germany, 1958), Vol. V, Part 1.

[11] See, for example, Källén's article on quantum electrodynamics (reference 10).

The first term is the vacuum expectation value of a product of free fields, and is thus equal to

$$\langle 0_{\text{out}}|\prod_{i=1}^{n}\phi_{\text{out}}(x_i)|0_{\text{out}}\rangle=\langle 0|\prod_{i=1}^{n}A(x_i)|0\rangle.$$

The remaining terms on the right each contain as a factor a quantity $\int dx_j'\Delta(x_j-x_j')j(x_j')$ for some x_j. This quantity vanishes unless $x_j\in V_{\pm}(\mathcal{R})$. Therefore, all of the terms on the right vanish except the first term when all x_j lie outside $V_{\pm}(\mathcal{R})$. For this case, we obtain

$$\langle\Psi|\prod_{i=1}^{n}A(x_i)|\Psi\rangle=\langle 0|\prod_{i=1}^{n}A(x_i)|0\rangle,$$

which is just (2).

The states (3) are thus localized states if the generating current operates in a confined region of space-time. They may be thought of as representatives of the type of state which occurs in a physical scattering process. The field of the apparatus which produces and accelerates the particles to be scattered is the external field $j(x)$. The particles then propagate from the region of production according to the free equations of motion. The apparatus for detecting scattered particles may also be thought of as an arrangement of fields and currents located in a bounded region of space-time. In the theoretical treatment of the scattering of fields, the initial and final states are usually idealized to one-particle states of definite momentum. Such states are not localized, and cannot be produced by apparatus confined to a bounded region of space-time. The localization associated with production and detection is accounted for in single-particle scattering theory by using a wave-packet description of initial and final states. It is in this sense that the localized states defined here provide a certain field theoretic analog of the wave-packet description.

4. LOCALIZED STATES IN TERMS OF NUMBER AND MOMENTUM EIGENFUNCTIONS

Having found that localized states may be generated by an external source interacting with the field within a bounded domain of space-time, we pass on to a further analysis of the structure of states satisfying the criterion of localization (2). For this purpose, we consider an expansion of the states in terms of the complete set associated with the momentum and number operators. Such an expansion has the form

$$|\Psi\rangle=\sum_{n=0}^{\infty}\sum_{\mathbf{k}_1\cdots\mathbf{k}_n}V^{-n/2}\psi_n(k_1,\cdots,k_n)|k_1,\cdots,k_n\rangle, \quad (4)$$

in which $|k_1,\cdots,k_n\rangle$ denotes a state containing n particles of momenta $k_1,\cdots,k_n$, and V denotes a finite volume of enclosure. This expansion is always possible for states of a free field. The four-vectors k_i must satisfy the conditions

$$k_i^2=\mathbf{k}_i^2-k_{i0}^2=-m^2, \quad k_{i0}>0, \quad (5)$$

the first in virtue of the field equations, and the second because of the physical requirement that the particles have positive energy. In addition, the coefficient $\psi_n(k_1,\cdots,k_n)$ must be symmetric in all its arguments in order to satisfy the Pauli principle. The states $|k_1,\cdots,k_n\rangle$ appearing in the expansion (4) may be obtained by applying the creation operators $a^{\dagger}(k)$ to the vacuum. They satisfy the normalization condition

$$\langle k_1,\cdots,k_n|k_1',\cdots,k_m'\rangle=\delta_{nm}\sum_{P(i)}\delta(\mathbf{k}_1-\mathbf{k}_{i1}')\cdots\delta(\mathbf{k}_n-\mathbf{k}_{in}), \quad (6)$$

the sum being taken over all permutations of the indices labelling the k'.

We retain the normalization (6) even in the case where two or more of the k are equal in order not to be forced to take these states into account separately in the calculations. This differs from the usual normalization by a factor $[\Pi(n_i!)]^{-1}$ if n_1, n_2, etc., k's are equal. In order to avoid singularities from these states because of products of δ functions with equal arguments appearing in (6), we have enclosed the system in a finite spatial volume V. Then the allowable values of k form a discrete manifold over which the summation in (4) is taken. The δ functions on the right of (6) are then to be interpreted as Kronecker symbols instead of Dirac δ functions. It may be shown that the quantities ψ_n are independent of V when we are dealing with localized states. In the final expressions for expectation values of products of fields, the volume may be taken infinite without these singularities reappearing, and we may then replace the summations by integrations:

$$\sum_{\mathbf{k}}=\int(d^3n/d^3k)d^3k=[V/(2\pi)^3]\int d^3k.$$

We have already introduced the creation operators $a^{\dagger}(k)$, in terms of which the field operator may be given as follows:

$$A(x)=\sum_{\mathbf{k}}\frac{1}{(2\omega V)^{\frac{1}{2}}}[a(k)e^{ikx}+a^{\dagger}(k)e^{-ikx}]$$

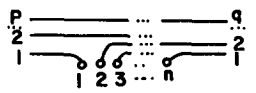
$$kx=\mathbf{k}\cdot\mathbf{x}-\omega x_0, \quad \omega=(\mathbf{k}^2+m^2)^{\frac{1}{2}}.$$

The normalization is chosen so that

$$[a(k),a^{\dagger}(k')]=\delta(\mathbf{k}-\mathbf{k}').$$

The quantities $\Psi(x_1,\cdots,x_n)=\langle\Psi|A(x_1)\cdots A(x_n)|\Psi\rangle-\langle 0|A(x_1)\cdots A(x_n)|0\rangle$ must vanish outside of a region $V_{\pm}(\mathcal{R})$ if (2) is to be satisfied. We wish to express this condition in terms of the $\psi_n(k_1,\cdots,k_n)$. This is greatly facilitated by the introduction of a graphical method of representating the matrix elements involved. From (4)

FIG. 1.



we obtain

$$|\Psi\rangle=\sum_{n=0}^{\infty}\sum_{k_1,\cdots,k_n}V^{-n/2}\psi_n(k_1,\cdots,k_n)a^\dagger(k_1)\cdots a^\dagger(k_n)|0\rangle$$

$$=\sum_{n=0}^{\infty}\sum_{k_1,\cdots,k_n}V^{-n/2}\psi_n(k_1,\cdots,k_n)(2^n\omega_1\cdots\omega_nV^{-n})^{\frac{1}{2}}$$

$$\times\int\cdots\int d^3x_1\cdots d^3x_n e^{ik_1x_1+\cdots+ik_nx_n}$$

$$\times A^{(-)}(x_1)\cdots A^{(-)}(x_n)|0\rangle.$$

$A^{(-)}(x)$ is the creation part of the field operator $A(x)$:

$$A^{(-)}(x)=\sum_{\mathbf{k}}\frac{1}{(2\omega V)^{\frac{1}{2}}}a^\dagger(k)e^{-ikx}.$$

We also list the following for future reference:

$$A^{(+)}(x)=A^{(-)\dagger}(x),\quad A(x)=A^{(+)}(x)+A^{(-)}(x),$$
$$[A^{(-)}(x),A^{(+)}(x')]=i\Delta^{(+)}(x-x'),$$
$$[A^{(+)}(x),A^{(+)}(x')]=[A^{(-)}(x),A^{(-)}(x')]=0.$$

On introducing the Fourier transform of $\psi_n(k_1,\cdots,k_n)$,

$$\psi_n(x_1,\cdots,x_n)=V^{-n}\sum_{k_1,\cdots,k_n}(2^n\omega_1\cdots\omega_n)^{\frac{1}{2}}$$

$$\times\psi_n(k_1,\cdots,k_n)e^{ik_1x_1+\cdots+ik_nx_n},\quad(7)$$

we have

$$|\Psi\rangle=\sum_{n=0}^{\infty}\int\cdots\int d^3x_1\cdots d^3x_n\psi_n(x_1,\cdots,x_n)$$

$$\times A^{(-)}(x_1)\cdots A^{(-)}(x_n)|0\rangle.\quad(8)$$

The expectation value of $A(x_1)\cdots A(x_n)$ becomes

$$\langle\Psi|A(x_1)\cdots A(x_n)|\Psi\rangle$$

$$=\sum_{p,q=0}^{\infty}\int\cdots\int d^3z_1\cdots d^3z_p d^3y_1\cdots d^3y_q$$

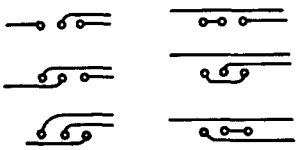
$$\times\psi_p^*(z_1,\cdots,z_p)\psi_q(y_1,\cdots,y_q)\langle0|A^{(+)}(z_p)\cdots$$

$$\times A^{(+)}(z_1)A(x_1)\cdots A(x_n)A^{(-)}(y_1)\cdots$$

$$\times A^{(-)}(y_q)|0\rangle.\quad(9)$$

The matrix element $\langle0|A^{(+)}(z_p)\cdots A^{(+)}(z_1)A(x_1)\cdots\times A(x_n)A^{(-)}(y_1)\cdots A^{(-)}(y_n)|0\rangle$ may be represented graphically as follows (Fig. 1). First, n points are drawn in a horizontal line, representing the n field operators $A(x_i)$. Then, lines representing particles are drawn, either connecting these points together in pairs, or extending to the left or right. Lines may also extend from left to right without touching one of the n points.

FIG. 2.



Exactly q lines must extend to the right and p lines to the left. The resulting figure describes the sequences of operations involved in computing the matrix element. The graphs, as well as the matrix elements are read from right to left. First the q creation operators $A^{(-)}(y_i)$ act on the vacuum to give a q-particle state. Then the field operators act in turn, either destroying a particle already present, or creating another particle. This results in a state of p particles, which are annihilated by the $A^{(+)}(z_i)$, giving the vacuum. Note that exactly one line emerges from each point of the graph. It may extend to the right or left representing destruction or creation, respectively, of the particle. If the figure cannot be drawn for the given values of n, p, and q, then the matrix element vanishes. It is usually possible to draw more than one graph for the given values of p, q, and n. For example, in the case $n-3$, $p-1$, $q-2$, 12 different diagrams are possible. Six of these are shown in Fig. 2. The other six are obtained by crossing the two lines extending to the right.[12]

The quantity

$$\langle0|A^{(+)}(z_p)\cdots A^{(+)}(z_1)A(x_1)\cdots\times A(x_n)A^{(-)}(y_1)\cdots A^{(-)}(y_q)|0\rangle\quad(10)$$

is represented by the sum of all these possible graphs. For a given graph, the line connecting a point P_1 to a point P_2 to the right of P_1 contributes a factor $i\Delta^{(+)}(P_1-P_2)$. We will not bother here to give a proof of all these statements, but only exhibit a set of formulas upon which an inductive proof may be based. These are simple consequences of (7).

$$\langle0|A^{(+)}(z_p)\cdots A^{(+)}(z_1)A(x_1)\cdots A(x_n)A^{(-)}(y_1)\cdots$$
$$\times A^{(-)}(y_q)|0\rangle$$
$$=\sum_{r=1}^{n}i\Delta^{(+)}(z_1-x_r)\langle0|A^{(+)}(z_p)\cdots A^{(+)}(z_2)A(x_1)\cdots$$
$$\times A(x_{r-1})A(x_{r+1})\cdots A(x_r)A^{(-)}(y_1)\cdots A^{(-)}(y_q)|0\rangle$$
$$+\sum_{r=1}^{q}i\Delta^{(+)}(z_1-y_r)\langle0|A^{(+)}(z_p)\cdots A^{(+)}(z_2)A(x_1)\cdots$$
$$\times A(x_n)A^{(-)}(y_1)\cdots A^{(-)}(y_{r-1})A^{(-)}(y_{r+1})\cdots$$
$$\times A^{(-)}(y_q)|0\rangle.$$

$$\langle0|A(x_1)\cdots A(x_n)A(y_1)\cdots A(y_q)|0\rangle$$
$$=\sum_{r=1}^{n}i\Delta^{(+)}(x_r-y_1)\langle0|A(x_1)\cdots A(x_{r-1})A(x_{r+1})\cdots$$
$$\times A(x_n)A^{(-)}(y_2)\cdots A^{(-)}(y_q)|0\rangle.$$

[12] These graphs are similar to Feynman graphs with only one line terminating at each vertex.

FIG. 3.

As an example of the graphical technique, we may consider the vacuum expectation value of a product of fields. This is the case $p=q=0$, so that no $A^{(+)}$ or $A^{(-)}$ operators appear in (10), and no lines extend to the right or left in the graphs. It is clear that the vacuum expectation value of an odd number of operators vanishes. We obtain

$$\langle 0|A(x_1)\cdots A(x_n)|0\rangle = \begin{cases} 0, & n \text{ odd} \\ \Sigma i^{n/2}\Delta^{(+)}(x_{i1}-x_{i2})\cdots\Delta^{(+)}(x_{in-1}-x_{in}), & n \text{ even,} \end{cases}$$

where the sum is taken over all possible ways to form $n/2$ pairs from the n points, preserving the original order of the x_i within each pair. In the case of 4-fields, the graphs are shown in Fig. 3.

When we are evaluating the quantity (7), many of the graphs will give the same contribution because of the symmetry of $\psi_n(x_1,\cdots,x_n)$. For example, in Fig. 2 only the graphs shown give an independent contribution. The six graphs obtained from them by permuting the terminal points of the lines at the right each give the same contribution to (9) as the corresponding unpermuted graphs. Another example is the normalization integral of the state (8): $\langle\Psi|\Psi\rangle=1$. The independent graphs are shown in Fig. 4. The contribution of each graph must be multiplied by $p!$, the number of possible graphs with crossed lines corresponding to the given graph. Figure 4 (a) is associated with the vacuum component of $|\Psi\rangle$. The resulting condition is

$$\sum_{p=0}^{\infty}\int\cdots\int dz_1\cdots dz_p dy_1\cdots dy_p p!\psi_p^*(z_1,\cdots,z_p) \times\psi_p(y_1,\cdots,y_p)i^p\Delta^{(+)}(z_1-y_1)\cdots\Delta^{(+)}(z_p-y_p)=1,$$

a further condition on the ψ_p.

We may now prove the following theorem:

No state of the form (4) or (8) can be localized if $\psi_n=0$ for all $n>N$, an arbitrary integer.

This means that a localized state must have an "infinite number of particles," i.e., there must be a nonzero probability of finding more than N particles in the state, however large N may be. The proof of this theorem depends upon the structure of expectation values of products of field operators, and on certain analyticity properties of these quantities considered as functions of the variables x_i. These analyticity properties follow from the spectral conditions (5).

FIG. 4.

Before giving the general proof, we prove the simpler theorem that a state of the form

$$|\Psi\rangle=\psi_0|0\rangle+\textstyle\sum_k V^{-\frac{1}{2}}\psi_1(k)|k\rangle,$$

with ψ_1 not identically zero, cannot be localized. This is a special case of the general theorem with $N=1$. Consider the quantity $\langle\Psi|A(x_1)A(x_2)|\Psi\rangle$. There are four diagrams contributing to this quantity, as shown in Fig. 5. The sum of (a) and (b) is $i\Delta^{(+)}(x_1-x_2)$ times the normalization integral of the state $|\Psi\rangle$, which is equal to unity. Thus the quantity $\Psi(x_1,x_2)$ which must be localized is the sum of (c) and (d):

$$\Psi(x_1,x_2)=i\int d^3z\psi_1^*(z)\Delta^{(+)}(z-x_1) \cdot i\int d^3y\Delta^{(+)}(x_2-y)\psi_1(y) +i\int d^3z\psi_1^*(z)\Delta^{(+)}(z-x_2) \cdot i\int d^3y\Delta^{(+)}(x_1-y)\psi_1(y)$$

$$=\Phi(x_1)\Phi^*(x_2)+\Phi(x_2)\Phi^*(x_1), \tag{11}$$

with

$$\Phi(x)=i\int d^3z\psi_1^*(z)\Delta^{(+)}(z-x) =-V^{-1}\sum_{\mathbf{k}}\psi_1(\mathbf{k})(2\omega)^{-\frac{1}{2}}e^{-ikx} =\int dk\Phi(k)e^{-ikx},$$

$$\Phi(k)=-(2\omega)^{\frac{1}{2}}(2\pi)^{-3}\theta(k)\delta(k^2+m^2)\psi_1^*(k). \tag{12}$$

We have expressed $\Phi(x)$, a solution of the Klein-Gordon equation, as a four-dimensional Fourier transform. The factors $\delta(k^2+m^2)$ and $\theta(k)$ are explicit expressions of the spectral conditions (5). If we replace x by the complex four-vector $z=x+iy$, expression (12) defines a function of the four complex components of z which is analytic for those values of z for which the integral converges. It is easily seen that this is the case in the region $y\in V_+$ and that $\Phi(z)$ is analytic in that region.

As we allow y to approach zero from the forward light cone, we obtain as a boundary value the function (12). If we now hold the space part of z fixed and real, we obtain a function $\Phi(\mathbf{x},z_0)$ which is analytic in the upper half of the z_0 plane, and takes on the value $\Phi(x)$ on the

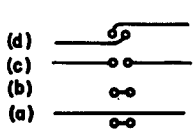


FIG. 5.

real axis. Then, by the uniqueness theorem for analytic functions, we conclude that if Φ vanishes in any time interval at the point x, then it vanishes for all time at that point.

By repeated application of this argument, we may prove that $\Phi(x)$ cannot vanish for all x outside $V_{\pm}(\mathfrak{R})$ unless it is identically zero, that is, unless $\psi_i(\mathbf{k})$ is identically zero. Let x_0 be a point outside the light cone for which $\Phi(x_0)\neq 0$. Then by (11),

$$\langle\Psi|A(x_0)A(x_0)|\Psi\rangle-\langle 0|A(x_0)A(x_0)|0\rangle=2|\Phi(x_0)|^2\neq 0,$$

violating the condition of localization (2), and completing the proof that no state of the form

$$|\Psi\rangle=\psi_0|0\rangle+V^{-\frac{1}{2}}\textstyle\sum_{\mathbf{k}}\psi_1(\mathbf{k})|\mathbf{k}\rangle$$

can be localized.

We now go on to the proof that no state of the form

$$|\Psi\rangle=\sum_{n=0}^{N}V^{-n/2}\psi_n(k_1,\cdots,k_n)|k_1,\cdots,k_n\rangle$$

can be localized. Consider the expectation value of the product of $2N$ field operators:

$$\langle 0|A(x_1)\cdots A(x_{2N})|0\rangle. \tag{13}$$

The diagrams contributing to this quantity may be classified as follows:

(A) diagrams in which no two points x_i are joined by a line,

(B_p) diagrams in which exactly $2p$ points x_i are joined by a line, $p<N$, and

(C) diagrams in which each x_i is joined to another x_i by a line.[13]

For example, some of the graphs for the case $N-2$ are shown in Fig. 6. Figures 6 (a), (b), and (c), are of type (C); (d) (e), (f), and (g) of type (B_1); and (h) and (j) of type (A). The sum of the contributions of all diagrams of type (C) to the quantity (13) is $\langle 0|A(x_1)\cdots A(x_n)|0\rangle$ time $\langle\Psi|\Psi\rangle$, or simply $\langle 0|A(x_1)\cdots A(x_n)|0\rangle$ since $|\Psi\rangle$ is a normalized state. Therefore, the sum of all contributions from diagrams of types (A) and (B_p) must vanish when all the x_i lie outside $V_{\pm}(\mathfrak{R})$. It is clear that a diagram of type (B_1) gives $i\Delta^{(+)}(x_i-x_j)$ times the sum of all contributions of type (A) to the quantity

$$\langle\Psi|A(x_1)\cdots A(x_{i-1})A(x_{i+1})\cdots \times A(x_{j-1})A(x_{j+1})\cdots A(x_{2N})|\Psi\rangle.$$

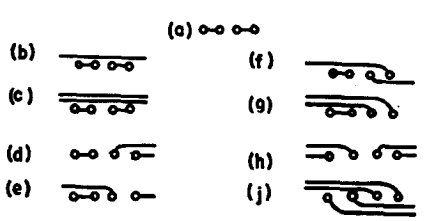
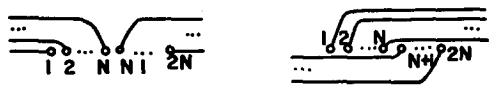


FIG. 6.

[13] We might also call this class B_N.

FIG. 7.



These contributions may be written

$$\begin{aligned}\langle\Psi|A(x_1)\cdots A(x_{i-1})A(x_{i+1})\cdots &\times A(x_{j-1})A(x_{j+1})\cdots A(x_{2N})|\Psi\rangle\\ -\langle 0|A(x_1)\cdots A(x_{i-1})A(x_{i+1})\cdots &\times A(x_{j-1})A(x_{j+1})\cdots A(x_{2N})|0\rangle\\ -\Psi^B(x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_{j-1},x_{j+1},\cdots,x_{2N}),&\end{aligned}$$

i.e., the quantity itself minus contributions of types (C) and (B). The term

$$\Psi^B(x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_{j-1},x_{j+1},\cdots,x_{2N})$$

may be in turn expressed as a sum of products of $\Delta^{(+)}$ functions with vacuum expectation values of products of a smaller number of fields. It is therefore clear that the (B)-type diagrams will contribute a quantity of the form

$$\begin{aligned}\sum_{i<j} i\Delta^{(+)}(x_i-x_j)[&\langle\Psi|A(x_1)\cdots A(x_{i-1})A(x_{i+1})\cdots\\ &\times A(x_{j-1})A(x_{j+1})\cdots A(x_{2N})|\Psi\rangle\\ &-\langle 0|A(x_1)\cdots A(x_{i-1})A(x_{i+1})\cdots\\ &\times A(x_{j-1})A(x_{j+1})\cdots A(x_{2n})|0\rangle]\\ &+\sum_{i<j,k<l} i\Delta^{(+)}(x_i-x_j)\cdot i\Delta^{(+)}(x_k-x_l)\\ &\times[\langle\Psi|\cdots|\Psi\rangle-\langle 0|\cdots|0\rangle]+\cdots,\end{aligned}$$

involving expectation values of products of less than $2N$ field operators. Now these quantities must all vanish when the x_i lie outside $V_{\pm}(\mathfrak{R})$ if $|\Psi\rangle$ is to be localized. For such x_i then, the only diagrams which need to be considered are those of type (A). Two diagrams of this type are shown in Fig. 7. The contribution of these diagrams is

$$\Phi(x_1,\cdots,x_N)\Phi^*(x_{N+1},\cdots,x_{2N})+\Phi^*(x_1,\cdots,x_N)\Phi(x_{N+1},\cdots,x_{2N}).$$

$$\begin{aligned}\Phi(x_1,\cdots,x_N)&=i^N\int\cdots\int d^3z_1\cdots d^3z_N\Delta^{(+)}(z_1-x_1)\cdots\\ &\quad\times\Delta^{(+)}(z_N-x_N)\psi_N^*(z_1,\cdots,z_N)\\ &=(-V)^{-N}\sum_{k_1,\cdots,k_N}(2^N\omega_1\cdots\omega_N)^{-\frac{1}{2}}\\ &\quad\times\psi_N^*(k_1,\cdots,k_N)e^{-ik_1x_1-\cdots-ik_Nx_N}.\end{aligned}\tag{14}$$

The other type (A) diagrams are obtained by rearranging the points x_i in (14). Note that for each diagram, there occurs another diagram which gives the complex conjugate of the original diagram, as in Fig. 6. If $|\Psi\rangle$ is to be localized, then it is necessary that the

quantity

$$\sum \Phi(x_{i_1},\cdots,x_{i_N})\Phi^*(x_{i_{N+1}},\cdots,x_{i_{2N}}), \qquad (15)$$

be localized in the sense of (2). The summation is taken over all arrangements of the x_i. Now, let us set $x_1=x_{N+1}$ and $x_2=x_3=\cdots=x_N=x_{N+2}=\cdots=x_{2N}$, and let x_2 be a fixed point outside $V_\pm(\mathfrak{R})$. Then (15) becomes

$$4N^2[(2N-2)!](N!)^{-2}|\Phi(x_1,x_2,\cdots,x_2)|^2 + 4N(N-1)[(2N-2)!](N!)^{-2} \times \mathrm{Re}\Phi^*(x_1,x_1,x_2,\cdots,x_2)\Phi(x_2,\cdots,x_2).$$

The symmetry of Φ in its N arguments has been used to obtain this expression. The first term arises from those diagrams in which the points x_1 and x_{N+1} are connected to opposite sides of the diagram, and the second term from those in which they are connected to the same side. The factors $4N^2[(2N-2)!](N!)^{-2}$ and $4N(N-1)[(2N-2)!](N!)^{-2}$ represent the total number of diagrams of each kind. In the second term, complex conjugate diagrams have been combined to give $\mathrm{Re}\Phi^*(x_1,x_1,x_2,\cdots,x_2)\Phi(x_2,\cdots,x_2)$.

The function $\Phi(x_1,x_2,\cdots,x_2)$ may be regarded as a function of x_1 if we hold x_2 fixed. Its Fourier transform with respect to x_1 vanishes except on the forward mass hyperboloid, $k_1^2=-m^2$, and $k_{10}>0$, by virtue of (14) and (5). It is therefore an analytic function in the components of $x_1\to z_1=x_1+iy_1$ in the region $y_1\in V_+$, the forward light cone. Thus, as before, there must be a point $x_1^{(0)}$ outside $V_\pm(\mathfrak{R})$ such that

$$\Phi(x_1^{(0)},x_2,x_2,\cdots,x_2)\neq 0.$$

There are two cases to consider:

(1) $\Phi(x_2,\cdots,x_2)=0$. Then (14) becomes

$$4N^2[(2N-2)!](N!)^{-2}|\Phi(x_1^{(0)},x_2,\cdots,x_2)|^2\neq 0,$$

and the state cannot satisfy the localization criterion; and

(2) $\Phi(x_2,\cdots,x_2)\neq 0$, where we have

$$\langle\Psi|A(x_2)\cdots A(x_2)|\Psi\rangle-\langle 0|A(x_2)\cdots A(x_2)|0\rangle = 2(2N)!(N!)^{-2}|\Phi(x_2,\cdots,x_2)|^2\neq 0.$$

In each case, we have been able to construct points x_i all outside of $V_\pm(\mathfrak{R})$ where $\langle\Psi|A(x_1)\cdots A(x_{2N})|\Psi\rangle \neq\langle 0|A(x_1)\cdots A(x_{2N})|0\rangle$. We have therefore proved that a state of the form

$$|\Psi\rangle=\sum_{n=0}^{N} V^{-n/2}\psi_n(k_1,\cdots,k_n)|k_1,\cdots,k_n\rangle$$

cannot satisfy criterion (2).

The theorem just proved allows us to confine our attention to states having an infinite number of particles in the sense explained above. If we attempt to discuss these states by the diagramatic method of this chapter, we find that an infinite number of graphs contribute to the expectation value of any given number of fields. For example, the sequence of graphs in Fig. 8 contributes to

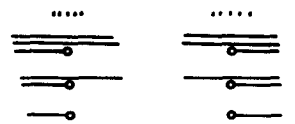


FIG. 8.

$\langle\Psi|A(x)|\Psi\rangle$. This quantity is, therefore, given by an infinite series, of which a typical term is

$$i\int\cdots\int d^3z_1\cdots d^3z_p d^3y_1\cdots d^3y_{p-1}\psi_p^*(z_1,\cdots,z_p) \times\psi_{p-1}(y_1,\cdots,y_{p-1})\Delta^{(+)}(z_p-x).$$

This series involves all the coefficients $\psi_p(z_1,\cdots,z_p)$ for all values of p. The condition that $\langle\Psi|A(x)|\Psi\rangle$ be localized is thus a very complicated condition on the ψ_p, collectively. There are similar conditions for $\langle\Psi|A(x_1)A(x_2)|\Psi\rangle$, etc. The only simplification that appears is that diagrams with two or more points joined together may be omitted. This is similar to the situation above with (B)-type diagrams. Since further analysis based on this method becomes very complicated, we pass to the approach described in the next two sections.

5. STATES OF THE FORM $\exp(iR)|0\rangle$, R LINEAR

Consider the state vector

$$\exp(iR)|0\rangle, \qquad (16)$$

obtained by applying an exponential operator to the vacuum.[14] The fact that such a state always satisfies our condition of localization when R is a Hermitian operator depending only on the field operator $A(x)$ with $x\in\mathfrak{R}$ follows from the property of local commutativity of the fields, and is therefore true in the case of interacting fields having this property as well as for free fields. Our further remarks in this section, however, apply to the free-field case. The precise form of R will be discussed later in this section, but for the present we may imagine R to be of the same form as (1), but with the integrations extending only over $\mathfrak{R}$. To insure the Hermiticity of R, the functions q_i of (1) must be real. Then since any point x_j outside $V_\pm(\mathfrak{R})$ is space-like with respect to the region R over which the integrals defining R are taken, local commutativity insures that $[R,A(x_j)]=0$. It is then easily shown that the state $\exp(iR)|0\rangle$ is localized, for

$$\langle 0|\exp(-iR)\prod_{j=1}^{n}A(x_j)\exp(iR)|0\rangle = \langle 0|\prod_{j=1}^{n}A(x_j)\exp(-iR)\exp(iR)|0\rangle = \langle 0|\prod_{j=1}^{n}A(x_j)|0\rangle$$

[14] The author is greatly indebted to R. Glaser (private communication to J. S. Toll) for calling these states to his attention.

because $\exp(iR)$ is a unitary operator. The fact that $\exp(iR)|0\rangle$ is a localized state considerably enlarges the manifold of such states at our disposal. One is led to conjecture that all localized states may be expressed in this form. Unfortunately, we have been able to neither prove nor to disprove this assertion. We will, therefore, confine ourselves here to as thorough an investigation as practicable of these states, indicating at the end some arguments supporting the conjecture.

First, it can be shown that the Poisson distribution state studied in Sec. 3 can be put into the form $\exp(iR)|0\rangle$, with

$$R=\int j(x)A(x)dx =\sum_k(2\omega)^{-\frac{1}{2}}(j^*(\mathbf{k},\omega)a(k)+j(\mathbf{k},\omega)a^\dagger(k)). \quad (17)$$

If we expand $|\Psi\rangle=\exp(iR)|0\rangle$ in momentum eigenstates, we find the expansion coefficient

$$(1/n!)\langle k_1,\cdots,k_n|\Psi\rangle = (i^n/n!)(2^n\omega_1\cdots\omega_n)^{-\frac{1}{2}}j(\mathbf{k}_1,\omega_1)\cdots j(\mathbf{k}_n,\omega_n) \times\exp(-\tfrac{1}{4}\textstyle\sum_k\omega^{-1}|j(\mathbf{k},\omega)|^2), \quad (18)$$

which is the same as that for the Poisson distribution state.

The property of R essential to give localization is that it involves only $A(x)$ with x restricted to $\mathfrak{R}$. Equation (17) is therefore not the only form possible for R. Another very natural form we might choose is the following:

$$R=\int d^3x g(x)\frac{\bar\partial}{\partial x_0}A(x) \equiv\int d^3x\left(g(x)\frac{\partial}{\partial x_0}A(x)-A(x)\frac{\partial}{\partial x_0}g(x)\right), \quad (19)$$

where $g(x)$ is a localized solution of the Klein-Gordon equation. The integral is taken over a space-like surface intersecting the region $\mathfrak{R}$. This surface usually will be chosen as a time plane $x_0=$constant, although this is, of course, not necessary. If we define the Fourier transform of $g(x)$ by

$$g(x)=\textstyle\sum_{\mathbf{k}}(2\omega V)^{-\frac{1}{2}}[g(\mathbf{k})e^{ikx}+g^*(\mathbf{k})e^{-ikx}],$$

we obtain

$$R=-i\textstyle\sum_{\mathbf{k}}[a(k)g^*(\mathbf{k})-a^\dagger(k)g(\mathbf{k})].$$

This is similar to the expression (17), and the expansion of $\exp(iR)|0\rangle$ into number and momentum eigenstates may be performed in the same manner. We find that the expansion coefficients agree with those of the Poisson distribution state (18), if we set

$$g(x)=\int dx'\Delta(x-x')j(x'),$$

which makes explicitly clear the fact that $g(x)$ is a localized solution of the Klein-Gordon equation, since $j(x)$ vanishes outside $\mathfrak{R}$.

We have investigated two different forms for the operator R appearing in (16) in the case where R is linear in $A(x)$. Both of these forms give rise to states of the Poisson distribution type discussed in Sec. 3. Each form may be generalized to involve products of more than one $A(x)$. The generalization of (17) is given by (1). The general form of (19) is

$$R=r_0+\int_{x_0=t}d^3x r_1(x)\frac{\bar\partial}{\partial x_0}A(x) +\iint_{x_{10}=x_{20}=t}d^3x_1d^3x_2r_2(x_1,x_2)\frac{\bar\partial}{\partial x_{10}}\frac{\bar\partial}{\partial x_{20}} \times A(x_1)A(x_2)+\cdots. \quad (20)$$

The $r_i(x_1,\cdots,x_n)$ as well as the $q_i(x_1,\cdots,x_n)$ of (1) can always be chosen as symmetric functions of their arguments.

Since we are looking for a general representation of localized states, we wish to decide which of these forms is the more suitable. This is accomplished by requiring that the representation be unique. In the case of the Poisson distribution state, we note that the expansion coefficients ψ_n determine the quantity $j(\mathbf{k},\omega)$ uniquely. This corresponds to the quantity $j(\mathbf{k},\omega)$ if we employ the four-dimensional integration (17), and to $g(\mathbf{k})$ if we use the three-dimensional integral (19) for R. Thus, $j(\mathbf{k},\omega)$ and $g(\mathbf{k})$ are fixed by the state. However, $j(\mathbf{k},\omega)$ is not sufficient to determine the function $j(x)$ appearing in (17), since it fixes only these Fourier components of $j(x)$ which satisfy the restriction $\mathbf{k}^2-k_0^2=-m^2$. The other Fourier components, off the mass shell, are entirely arbitrary inasmuch as they may be varied in any way without affecting the state (16). In fact $j(x)$ need not even vanish outside $\mathfrak{R}$ in order that (16) be localized. It is only necessary that a localized function $j(x)$ exists having the Fourier components $j(\mathbf{k},\omega)$ on the mass shell. On the other hand, the function $g(x)$ appearing in the expression (19) for R is a solution of the Klein-Gordon equation, and is therefore completely determined by its Fourier components on the mass shell. It is thus fixed uniquely once the expansion coefficients (18) of the state are given. For this reason, we will use the form (19) and its generalization (20) for the operator R appearing in (16).

The nonuniqueness of the four-dimensional form can be understood from a different point of view. Equation (17) may be regarded as a linear combination of the field operators at points in the region $\mathfrak{R}$. A linear combination of quantities is unique only if the quantities are linearly independent. The field operators in our extended region of space-time, however, are not linearly independent, since an operator at a given time x_0 can

be expressed through the equation of motion in terms of operators at another time x_0'. Since only points within the light cone contribute, we have a linear relation involving only field operators taken at points of $\mathcal{R}$. In the other case, the integral (19) involves only field operators and their time derivatives at a given time. These are dynamically independent quantities, and therefore no linear relation between them exists. This explains why we obtain a unique representation with (19) but not with (18).

6. STATES OF THE FORM $\exp(iR)|0\rangle$, GENERAL

Having discussed states of the form (16) with R linear in $A(x)$, we now pass to a consideration of more complicated forms of R. The general form of R is taken to be (20), with $r_n(x_1,\cdots,x_n)$ a localized solution of the Klein-Gordon equation symmetric in all of its arguments. The expectation value

$$\langle\Psi|A(x_1)\cdots A(x_n)|\Psi\rangle = \langle 0|\exp(-iR)A(x_1)\cdots A(x_n)\exp(iR)|0\rangle$$

may be expressed in the form

$$\langle 0|\Gamma(x_1)\cdots\Gamma(x_n)|0\rangle, \tag{21}$$

with

$$\Gamma(x)=\exp(-iR)A(x)\exp(iR)$$
$$=A(x)+\exp(-iR)[A(x),\exp(iR)]$$
$$=A(x)+\sum_{n=1}^{\infty}\frac{i^n}{n!}[[\cdots[A(x),R],R]\cdots], \tag{22}$$

where the nth term contains the n-fold commutator of $A(x)$ with R. Thus, the expectation value of a product of field operators $A(x)$ in a state $|\Psi\rangle$ reduces to the vacuum expectation value of the product of Γ operators (21). $\Gamma(x)$ is given in terms of $A(x)$ by the infinite series (22). The exact form of $\Gamma(x)$ will depend upon the functions $r_n(x_1,\cdots,x_n)$ which define R. Suppose that $r_n=0$ for $n>N$, i.e., R is a sum of terms in each of which at most N operators $A(x)$ appear. We will express this by writing $R=0(A^N)$. It is easily seen that the commutator of an operator of order p with an operator of order q is an operator of order $p+q-2$. It is clear then that the nth term of the series (22) will be of order

$$(1+N-2)+(N-2)+\cdots+(N-2)=1+n(N-2).$$

Thus, if $N>2$, the operator $\Gamma(x)$ is of infinite order in $A(x)$. For $N=1$, the $n=0$ term, which is just $A(x)$, is of order 1, and the term $n=1$ is of order zero. The remaining terms vanish. For $N=2$, each term of the series is of order 1. These two cases are therefore particularly simple. The first, $N=1$, is the Poisson distribution state which we have already discussed at length. For this state, $\Gamma(x)$ is given by

$$\Gamma(x)=A(x)+r_1(x), \tag{23}$$

where the c number $r_1(x)$ is to be identified with the function $g(x)$ of Eq. (19). For the case $N=2$, we take R to be

$$R=\int\int d^3x_1 d^3x_2 r(x_1,x_2)\frac{\bar\partial}{\partial x_{10}}\frac{\bar\partial}{\partial x_{20}}A(x_1)A(x_2). \tag{24}$$

By making use of the fact that $r(x_1,x_2)$ is a symmetrical solution of the Klein-Gordon equation, and therefore satisfies the following equation:

$$r(x_1,x_2)=-\int d^3x_1'\Delta(x_1-x_1')\frac{\bar\partial}{\partial x_{10}'}r(x_1',x_2),$$

we obtain

$$[A(x),R]=2i\int\int d^3x_1 d^3x_2 r(x_1,x_2)\frac{\bar\partial}{\partial x_{10}}\frac{\bar\partial}{\partial x_{20}}$$
$$\times A(x_1)\Delta(x-x_2)$$
$$=2i\int d^3x_1 r(x,x_1)\frac{\bar\partial}{\partial x_{10}}A(x_1),$$

where we have also used the commutation relation $[A(x),A(x')]=i\Delta(x-x')$. The n-fold commutator $[[\cdots[A(x),R],R]\cdots]$ may be computed similarly:

$$[[\cdots[A(x),R],R],\cdots]$$
$$=(2i)^n\int\cdots\int d^3x_1\cdots d^3x_n r(x,x_1)\frac{\bar\partial}{\partial x_{10}}$$
$$\times r(x_1,x_2)\frac{\bar\partial}{\partial x_{20}}\cdots r(x_{n-1},x_n)\frac{\bar\partial}{\partial x_{n0}}A(x_n).$$

We therefore find that

$$\Gamma(x)=A(x)+\int d^3x'\gamma(x,x')\frac{\bar\partial}{\partial x_0'}A(x'), \tag{25}$$

$$\gamma(x,x')=\sum_{n=1}^{\infty}\frac{(-2)^n}{n!}\int\cdots\int d^3x_1\cdots d^3x_n r(x,x_1)\frac{\bar\partial}{\partial x_{10}}$$
$$\times r(x_1,x_2)\cdots\frac{\bar\partial}{\partial x_{n-1,0}}r(x_{n-1},x'), \tag{26}$$

by substituting these expressions into (22). For $N>2$, it is also possible to give the form of $\Gamma(x)$, but we will not do this here.

There is a strong similarity of the relation between $A(x)$ and $\Gamma(x)$ to that between $A_{\text{in}}(x)$ and $A_{\text{out}}(x)$ in a relativistic field theory with interaction. The correspondence is exact if we allow interaction with an external source. This is easily seen from the example given in Sec. 3. The relation

$$\Gamma(x)=\exp(-iR)A(x)\exp(iR)$$

is analogous to the relation

$$A_{\text{out}}(x)=S^{\dagger}A_{\text{in}}(x)S.$$

A relation of the form (22) also exists between in- and out-going fields if we introduce the phase matrix η by $S=\exp(-i\eta)$. There is one point of dissimilarity between $\Gamma(x)$ and the outgoing field of a system such as that described by the quantum electrodynamics where no external current acts. There, the outgoing field is invariant under displacements, whereas in our case the localized nature of the operator R destroys the displacement invariance of $\Gamma(x)$ considered as a free field. Thus the vacuum expectation value of a product of Γ operator does not have the simple properties of the Wightman functions.[2] They are not, for example, functions of the coordinate differences only, and do not have the same simple analyticity properties. It is still possible, however, to draw certain conclusions by means of analyticity properties, as we shall do below in the case where Γ has the form (24).

Before proceeding further, we will note that, since $\Gamma(x)$ is obtained from $A(x)$ by the unitary transformation $\exp(-iR)A(x)\exp(iR)$, it must satisfy the same commutation relation: $[\Gamma(x),\Gamma(y)]=i\Delta(x-y)$. In the case where R is linear in $A(x)$, $N=1$ above, this can also be seen directly from the expression (23) for $\Gamma(x)$, since $r_1(x)$ is a c-number function and thus commutes with $A(x)$. For the case $N=2$, where R has the form (24) and $\Gamma(x)$ the form (25), we find

$$[\Gamma(x),\Gamma(y)]=i\Delta(x-y)-i\gamma(x,y)+i\gamma(y,x)+i\int d^3z\gamma(x,z)(\bar{\partial}/\partial z_0)\gamma(z,y).$$

It is therefore necessary that $\gamma(x,y)$ satisfy the relation

$$\gamma(x,y)-\gamma(y,x)-\int d^3z\gamma(x,z)(\bar{\partial}/\partial z_0)\gamma(z,y)=0.$$

This is not an independent relation, but follows identically from (26).

It is clear that the condition of localization places certain restrictions upon the region of space where $\gamma(x,y)$ may differ from zero. It is our purpose to find out what these restrictions are and to find the corresponding conditions for the function $r(x,y)$ in terms of which $\gamma(x,y)$ is defined. Let us assume that on the reference time plane over which the integral (25) is taken, $\gamma(x,y)$ and its derivatives with respect to x_0 and y_0 and its mixed second derivative with respect to x_0 and y_0 all vanish when either x or y lies outside a large but bounded region $\mathcal{S}$ of 3-space. These four quantities, $\gamma(x,y)$ and its 3-time derivatives, when given for all space at a given time, completely determine $\gamma(x,y)$ for all space-time, since they provide the initial values from which $\gamma(x,y)$ may be computed from the Klein-Gordon equation. The region $\mathcal{S}$ is arbitrary, and may be taken much larger than $\mathcal{R}$, the region of localization.

From (25) we obtain

$$\begin{aligned}\langle 0|\Gamma(x)\Gamma(y)|0\rangle&=\langle 0|A(x)A(y)|0\rangle\\&+i\int d^3y'\gamma(y,y')\frac{\bar{\partial}}{\partial y_0'}\Delta^{(+)}(x-y')\\&+i\int d^3x'\gamma(x,x')\frac{\bar{\partial}}{\partial x_0'}\Delta^{(+)}(x'-y)\\&+i\int\int d^3x'd^3y'\gamma(x,x')\gamma(x,y')\frac{\bar{\partial}}{\partial x_0'}\frac{\bar{\partial}}{\partial y_0'}\\&\times\Delta^{(+)}(x'-y'),\end{aligned}\tag{27}$$

where we have used the relation

$$\langle 0|A(x)A(y)|0\rangle=i\Delta^{(+)}(x-y).$$

The sum of the last three terms on the right side of (27) must vanish when both x and y are outside $V_{\pm}(\mathcal{R})$. We denote these terms by a, b, and c, respectively. The Fourier transform of the singular function $\Delta^{(+)}$ appearing in (27) vanishes except on the forward mass hyperboloid $k^2=-m^2$, and $k_0>0$. Therefore, as in Sec. 4, it is an analytic function of the components of the vector x whenever the imaginary part of x lies in the forward light cone. We conclude that the quantity a is analytic in x, and the quantity b is analytic in y. The argument of Sec. 4 can therefore be applied to show that if these functions vanish over any extended region of space-time in their analytic arguments, they must vanish everywhere.

The localization condition requires that the quantity $a+b+c$ together with its three time derivatives, vanish outside the union of the regions $\mathcal{R}\times\mathcal{U}$ and $\mathcal{U}\times\mathcal{R}$. $\mathcal{R}\times\mathcal{U}$ is the cartesian product of the region $\mathcal{R}$ and the whole space,[15] denoted here by $\mathcal{U}$. In consequence of our assumption that $\gamma(x,y)$ vanishes outside of the region $\mathcal{S}\times\mathcal{S}$, it is evident from (27) that $a=0$ outside of $\mathcal{U}\times\mathcal{S}$, that $b=0$ outside of $\mathcal{S}\times\mathcal{U}$, and that $c=0$ outside of $\mathcal{S}\times\mathcal{S}$. Now, consider the region $(\mathcal{U}-\mathcal{S})\times(\mathcal{S}-\mathcal{R})$, shown schematically as the shaded region in Fig. 9.

In this region $a+b+c$ must equal zero. But we have shown that b and c vanish there. Thus, a must also vanish. We can conclude from this by means of the analyticity of a in the variable x that a vanishes for all x when y is in $(\mathcal{S}-\mathcal{R})$. We may, therefore, subtract the region $\mathcal{U}\times(\mathcal{S}-\mathcal{R})$ from the region we have already found, and obtain the result that $a=0$ outside of $\mathcal{U}\times\mathcal{R}$. By an entirely similar argument, $b=0$ outside of $\mathcal{R}\times\mathcal{U}$. The localization condition may now be applied to show that $c=0$ outside the union of $\mathcal{S}\times\mathcal{R}$ and $\mathcal{R}\times\mathcal{S}$. The

[15] In this section, we use $\mathcal{R}$ to denote the three-dimensional region over which the integrals in (5.7) extend, and not the four-dimensional space-time region used previously.

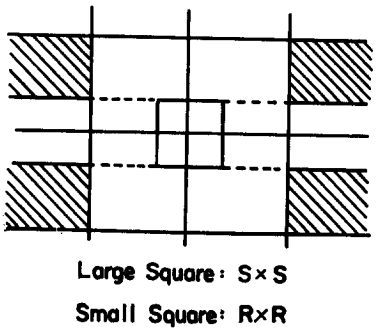


FIG. 9.

same results apply to the three time derivatives of each of these quantities.

It is possible to find a restriction on the region where $\gamma(x,y)$ may differ from zero from the above. From (27) we obtain

$$\mathrm{Im}a=\tfrac{1}{2}\int d^3y'\gamma(y,y')\frac{\bar{\partial}}{\partial y_0'}\Delta(x-y'). \tag{28}$$

Here, we have used $\Delta^{(+)}(x)=\frac{1}{2}(\Delta(x)-i\Delta^{(1)}(x))$. It follows immediately from this and the properties of the Δ function that $\mathrm{Im}a=-\frac{1}{2}\gamma(y,x)$. The corresponding relation between the three time derivatives may be derived in the same way after differentiating both sides of (28). Taking into account that Ima and its derivatives vanish outside $\mathcal{U}\times\mathcal{R}$, we find

$$\gamma(x,y)=\frac{\partial\gamma(x,y)}{\partial x_0}=\frac{\partial\gamma(x,y)}{\partial y_0}=\frac{\partial^2\gamma(x,y)}{\partial x_0\partial y_0}=0,$$
$$\text{if } (x,y)\notin\mathcal{R}\times\mathcal{U}. \tag{29}$$

Equation (29) implies that $\gamma(x,y)$ for arbitrary y is localized in x in the region $V_\pm(\mathcal{R})$. Our argument shows that (29) is a necessary condition that the expectation value of the product of two fields satisfy the localization criterion. That it is also sufficient is obvious from Eq. (27). Note that we may add a linear term of the form (19) to the expression (24) for R without interfering with the above argument. In fact, the localization condition for $\langle 0|\Gamma(x)|0\rangle$ requires that $g(x)$ and $\partial g(x)/\partial x_0$ be confined to the region $\mathcal{R}$, so that it does not enter into the above considerations.

It has now been shown that condition (29) is equivalent to localization of the state $\exp(iR)|0\rangle$, where R is given by (24). Is it possible to conclude from this that $r(x,y)$ and its time derivatives are zero outside of $\mathcal{R}\times\mathcal{R}$? It is obvious from (26) that this property of $r(x,y)$ is sufficient to guarantee that $\gamma(x,y)$ satisfies (29). Furthermore, if we assume only that $r(x,y)$ is confined, for example, to the cross-shaped region $\mathcal{U}\times\mathcal{R}+\mathcal{R}\times\mathcal{U}$, (29) can only be satisfied if a large-scale cancellation occurs among the terms of the series (26) for $\gamma(x,y)$ at points x lying outside of $\mathcal{R}$. It seems unlikely that this is possible, although we have not been able to prove its impossibility. It is shown in the Appendix that Eq. (26) may be inverted and $r(x,y)$ expressed in terms of $\gamma(x,y)$:

$$r(x,y)=\sum_{n=1}^{\infty}\frac{(-1)^n}{2n}\int\cdots\int d^3x_1\cdots d^3x_{n-1}$$
$$\times\gamma(x,x_1)\frac{\bar{\partial}}{\partial x_{10}}\cdots\frac{\bar{\partial}}{\partial x_{n-1,0}}\gamma(x_{n-1},y). \tag{30}$$

In order to derive (30), certain very restrictive and ad hoc assumptions must be made about the function $r(x,y)$. We cannot, therefore, claim general validity for this relation. However, if these assumptions are satisfied, it is clear from (30) that $r(x,y)$ with its time derivatives must be confined to $\mathcal{R}\times\mathcal{R}$. For, in consequence of conditions (29), each term of the series (30) is equal to zero when x lies outside $\mathcal{R}$, and so also their sum. Then, from the symmetry of $r(x,y)$ in x and y, we conclude that it also vanishes when y is not in $\mathcal{R}$.

We will now give a summary of the results of the last two sections. Our aim in considering states of the form (16) was to provide a general representation of states satisfying the definition of localization (2). Two different forms of the Hermitian operator R were considered, and the first rejected because the corresponding representation proved to be nonunique. The second form of R is determined by a sequence of functions $r_n(x_1,\cdots,x_n)$, each of which is a solution of the Klein-Gordon equation independently in each argument. We have shown that when R is linear in the field, the function $r_1(x)$ must be localized in the region $V_\pm(\mathcal{R})$. When R is quadratic in the fields, we proved that $r_2(x,y)$ must be zero outside of $V_\pm(\mathcal{R})\times V_\pm(\mathcal{R})$, subject to certain assumptions on the nature of $r_2(x,y)$.

In order to give a complete proof of the generality of the representation (16), we would have to show (a) that any state can be written in the form $\exp(iR)|0\rangle$ if $r_n(x_1,\cdots,x_n)$ is not restricted to be confined to any particular region of space, (b) that the localization condition then requires that $r_n(x_1,\cdots,x_n)$ vanish outside $V_\pm(\mathcal{R})\times\cdots\times V_\pm(\mathcal{R})$, and (c) that the representation is unique. The results of this section and the last show that (b) and (c) are true in some simple cases. Although it has not been possible here to give a complete general proof, it is encouraging to find that the conjecture that (16) is a general representation of localized states is substantiated in those cases in which it could be verified.

7. DISCUSSION AND CONCLUSIONS

In this paper, we have introduced a definition of localization based on taking the field variables as the primary measurable quantities. This enables us to avoid some of the difficulties which occur when one formulates such a definition in terms of particle observables. Our definition is completely Lorentz invariant, since it is formulated in terms of Lorentz-invariant quantities and refers to the situation of the field over all space-time. Two states localized in different

regions of space are not expected to be orthogonal because the two regions of localization $V_\pm(\mathcal{R}_1)$ and $V_\pm(\mathcal{R}_2)$ always overlap, however far $\mathcal{R}_1$ may be removed from $\mathcal{R}_2$. It may be verified that two states of the Poisson distribution type discussed in Chap. III are not orthogonal. Our definition thus seems satisfactory and appropriate to the current formulations of quantum field theory.

We have given results concerning the nature of the states satisfying the localization condition. It was seen in Sec. 5 that there is a large class of such states expressible in the simple form $\exp(iR)|0\rangle$. We have given arguments indicating that *any* localized state may be represented in this form, and proposed a general method of proof. While we have concentrated our attention in a large part of this thesis to the free-field case, this representation offers promise of being valid and general for the case of interacting fields as well.

Our motivation in taking up this problem was the hope that the results might be useful as the basis for a general approach to scattering problems in field theory. It does indeed appear that the states satisfying our definition of localization do provide an idealized description of the initial and final states of scattering processes which is nearer to the physical reality of production and detection than the customary idealization of single particle states. It is possible that by applying the methods of this paper, some of the difficulties associated with the asymptotic condition might be clarified. It is even possible that a description of scattering experiments might be given without appealing to the asymptotic condition. Such a description would necessarily be more complicated in some respects. For example, the theory of Sec. 4 shows that localized states may not contain a finite number of particles. This requires that any description of scattering in terms of localized states involve an infinite number of S-matrix elements between states of definite particle number.

However, it is easily seen from our example in Sec. 3 that if the current $j(x)$ is weak, we obtain a localized state in which the vacuum amplitude dominates, and the one-particle amplitude is much larger than the remaining amplitudes. It may, therefore, be possible to isolate the one-particle scattering terms from the others by a limiting process in which the strength of the current approaches zero.

It is not possible at present to evaluate completely the merits and defects of such a procedure, but it is hoped that the results of this paper will provide the foundation of a useful alternative to the conventional description of the scattering process in quantum field theory.

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APPENDIX: INVERSION OF EQUATION (26)

In order to invert Eq. (26), we introduce a complete orthonormal set $\{\phi_i(x)\}$ of solutions of the Klein-Gordon equation:

$$\int d^3x\phi_i^*(x)\overleftrightarrow{\frac{\partial}{\partial x_0}}\phi_j(x)=\sigma_i\delta_{ij},$$
$$\sum_i \sigma_i\phi_i(x)\phi_i^*(y)=i\Delta(x-y). \tag{A1}$$

$\sigma_i=\pm1$ according to whether $\phi_i(x)$ belongs to positive or negative energy. We define the quantity r_{ij} by

$$r_{ij}=\int\int d^3xd^3y\phi_i^*(x)\overleftrightarrow{\frac{\partial}{\partial x_0}}r(x,y)\overleftrightarrow{\frac{\partial}{\partial y_0}}\phi_j(y),$$
$$r(x,y)=-\sum_{i,j}\sigma_i\sigma_j r_{ij}\phi_i(x)\phi_j^*(y). \tag{A2}$$

The second equation (A2) follows from the first upon making use of (A1). The reality and symmetry of $r(x,y)$ require that $r_{ij}=r_{ji}^*$. γ_{ij} is defined in a similar way in terms of $\gamma(x,y)$, but $\gamma_{ij}\neq\gamma_{ji}^*$ because $\gamma(x,y)$ is not symmetric.

We now choose the set $\{\phi_i(x)\}$ in such a way that r_{ij} becomes a diagonal matrix, and set $r_{ij}=r_i\delta_{ij}$. After introduction of the newly defined quantities, (26) is transformed into

$$i\sigma_i\gamma_{ij}=(e^{-2i\sigma_i r_i}-1)\delta_{ij}.$$

Thus, γ_{ij} is also diagonal, with eigenvalues

$$\gamma_i=(1/i\sigma_i)(e^{-2i\sigma_i r_i}-1).$$

This equation may be solved for r_i, yielding

$$-2i\sigma_i r_i=\log(1+i\sigma_i\gamma_i). \tag{A3}$$

The quantities γ_i are complex numbers lying on a circle of unit radius and center at the point $+1$. If r_i is such that γ_i lies within the unit circle centered at the origin, then (A3) may be expanded into a convergent power series in γ_i:

$$2i\sigma_i r_i=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(i\sigma_i\gamma_i)^n.$$

If this expansion is possible for all r_i, then we may reintroduce $r(x,y)$ and $\gamma(x,y)$ by means of (A2) and obtain Eq. (30):

$$r(x,y)=\sum_{n=1}^{\infty}\frac{(-1)^n}{2n}\int\cdots\int d^3x_1\cdots d^3x_{n-1}$$
$$\times\gamma(x,x_1)\overleftrightarrow{\frac{\partial}{\partial x_{10}}}\cdots\overleftrightarrow{\frac{\partial}{\partial x_{n-1,0}}}\gamma(x_{n-1},y).$$