Outline of a History of differential Geometry (*)

(II)

6. — Gauss

If the French drew the most consistent economic and political consequences from their revolution, the Germans on the other hand were more profoundly stimulated by the accompanying intellectual upheaval. France had NAPOLEON, but Germany had BEETHOVEN, KANT and GAUSS. The small town philosopher of Königsberg and the small town mathematician of Göttingen represent as truly the essential aspects of the new era as the little corporal himself.

Germany, after the death of LEIBNIZ, had participated only in the construction of new mathematical ideas in so far as its despots had imported genius to adorn their throne. The same causes that bring the revolution to France revivify the German intellect. Its clearest expression in mathematics is CARL FRIEDRICH GAUSS (1777-1855).

For us the activity of GAUSS is threefold : as inventor of non-Euclidean geometry, as inventor of intrinsic differential geometry, and as a theoretical geodesist. This geodetical work, as the practical one, was at the bottom of all his geometrical discoveries. Most of them were made in the period between 1815 and 1830. From 1821 to '25 GAUSS was busy surveying the Kingdom of Hanover.

We shall not discuss GAUSS' work on non-Euclidean geometry, though it profoundly affected later differential geometry. It has been done in considerable detail. We shall deal only very

(*) See *Isis*, 19, 92-120.

briefly with his geodetical papers. Our main subject has to be GAUSS' direct intervention into the theory of surfaces.

GAUSS' mathematical investigation of the problem of map projection was written in 1822 under the title " Allgemeine Auflösung der Aufgabe die Theile einer gegebnen Fläche auf einer andren gegebnen Fläche so abzubilden dass die Abbildung dem Abgebildeten in den kleinsten Theilen ähnlich wird." (1) It is the general theory of the conformal representation of arbitrary surfaces, and it represents a continuation of papers by EULER and LAGRANGE. As EULER and LAGRANGE, GAUSS uses " Gaussian coordinates," taking x, y, z, equal to functions of t and u, but where EULER and LAGRANGE represent a surface only on a plane, GAUSS represents two arbitrary surfaces upon each other. The solution, which uses complex functions, is applied to different cases, for instance, that of mapping a rotation ellipsoid on a sphere.

Later GAUSS came back to this subject and sought that conformal representation of the rotation ellipsoid on a sphere which is best for certain purposes. (2)

This work led GAUSS to the general theory of surfaces. A manuscript on this subject dates from 1825. He published a revised paper on this theory in 1827. (3)

These *Disquisitiones generales circa superficies curvas* of 1827 mean an entire departure from the French methods. (4) A new line of approach is found in the investigation of the intrinsic properties of surfaces, depending only on the linear element, and not the properties of the surface as imbedded in Euclidean three-space. Such intrinsic properties are unchanged by bending

(1) Printed in *Astronomische Abh.* 1825. See GAUSS Werke IV, p. 189-216. The paper was a solution to a problem posed by the Academy at Copenhagen in 1822.

(2) C. F. GAUSS, Untersuchungen über höhere Geodäsie, I, II, *Göttingen Abh.* 2 (1844), 3 (1847), GAUSS' Werke IV, p. 259-300, 301-340.

(3) The older ms. has the title : " Neue allgemeine Untersuchungen über die krummen Flächen " (1825), Werke VIII, p. 408 (with remarks of STÄCKEL). The " Disquisitiones " were published in *Commentationes soc. reg. sc. Göttingensis*, 6 (1828), Werke IV, p. 217-258. English translations of both papers by J. C. MOREHEAD and A. M. HILTEBEITEL. Princeton Univers. Library, 1902, p. 126, with bibliography of papers using GAUSS' method, and many notes.

(4) That GAUSS was acquainted with MONGE's contributions to geometry is shown by his favorable review of MONGE's " Géométrie descriptive," *Gött. gel. Anzeiger* 1813, Werke IV, p. 359-360.

the surface without tearing or stretching. GAUSS writes the linear element

$$ds = \sqrt{(Edp^2 + 2Fdpdq + Gdq^2)},$$

where p and q are "Gaussian" curvilinear coordinates on the surface. The main theorem is that the quantity $k = \frac{1}{R_1 R_2}$, a result from EULER's paper of 1760, depends only on E, F, G, and their first and second derivatives. GAUSS proves this "theorema egregium" in more than one way, first by taking the general linear element, and then by assuming $ds^2 = dp^2 + Gdq^2$. This involves a discussion of geodesics, and the discovery of another outstanding theorem, the one dealing with the sum of the angles in a geodesic triangle. This theorem is one of the first examples of so-called differential geometry in the large, and was partly anticipated by the corresponding theorem of LEGENDRE on the sum of the angles of a spherical triangle. (5) GAUSS proves the theorem by a transformation of the surface integral of curvature k, the "curvatura integra," into a line integral. The paper ends with expansions in series of different functions on the surface, in a coordinate system $ds^2 = dp^2 + G\,dq^2$.

These results constitute the essentials of the intrinsic theory of surfaces, linear element, geodesics, Gaussian curvature, curvatura integra, and geodesic triangle. Later investigations have not been able to add many more essential elements; they have shown only that with linear element and Gaussian curvature the theory is complete. One simple conception seems to be lacking, that of geodesic curvature, but in GAUSS' unpublished paper of 1825 this also appears, as "Seitenkrümmung."

GAUSS makes much use of the spherical representations of a surface, an idea he derived from astronomy, and indeed he defines the curvature k as the limit of the quotient of a small area of the surface and the corresponding area of the unit sphere. This raises, perhaps, the question of GAUSS' originality. Spherical representation, geodesics, curvilinear coordinates are all to be found in EULER—why admire GAUSS for their discovery? RODRI-

(5) A. M. LEGENDRE, *Mem. div. sav.* 1787, p. 358; also in the Appendix, Section 5, to his trigonometry. LEGENDRE's theorem was known to GAUSS Disquisitiones, Sec. 27.

GUES even had found the ratio between the surface elements of surface and unit sphere. But, though they were known to EULER, nobody had sensed their importance before GAUSS, and that after fifty years of productivity in France. And the light that GAUSS throws on both old and new conceptions as intrinsic properties of the surface was altogether new.

The invariance of the curvature under bending has been found in papers of GAUSS dating back to 1822. In these papers he uses the linear element of his paper on map projection, $ds^2 = m^2 (dt^2 + du^2)$.

However, nothing in the *Disquisitiones*, as KLEIN remarks, (6) shows that GAUSS kept his boldest ideas to himself. His manuscripts show that he was in possession of the non-Euclidean geometry when writing his paper on surfaces, but he did not hint at it even remotely. But it throws a new and better light upon his theorem on the sum of the angles of a geodesic triangle. It was a theorem enabling the physicist to test the nature of physical space. It was a step in the direction later consistently followed by RIEMANN and HELMHOLTZ. GAUSS indeed measured geodesical triangles, but did not find confirmation of his belief that there might be a perceptible curvature (negative, as GAUSS thought, in agreement with the later theories of LOBATSCHEWSKY and BOLYAI) of space.

Of GAUSS' contribution to notation and nomenclature we mention the symbols E, F, G, D, D', D'' for what we now call the coefficients of the first and second fundamental differential form, and the word " conformal." (6a)

GAUSS became the teacher of the entire learned world, but he created no direct school as MONGE had done. Conditions in Germany were different from those in France. The only exception, so far as differential geometry is concerned, was FERDINAND MINDING (1806-1885) in far-away Dorpat, who filled the

(6) F. KLEIN, Vorlesungen über die Entwicklung d. Mathem. I, p. 16. Here are excellent descriptions of the work of GAUSS.

(6a) In the first paper on higher geodesy, 1844 : " ich werde daher dieselben conforme Abbildungen oder Übertragungen nennen, in dem ich diesem sonst vagen Beiworte eine mathematisch scharf bestimmte Bedeutung beilege," Werke IV, p. 262. The word is, indeed, already used by F. T. SCHUBERT, " De projectione sphaeroidis ellipticae geographica," *Nova Acta Petr.*, p. 130-146, see CANTOR IV, p. 575.

gap left by GAUSS and published a paper on the geodesic curvature (6b). Some years later MINDING published other papers of outstanding importance, all showing a deep understanding of GAUSS' ideas. In them he took up the " problem of MINDING," that is the question of the applicability of surfaces on each other. One of his theorems states that two surfaces of constant curvature are always applicable. (6c) He determined the rotation surfaces of constant positive curvature and established the existence of the helicoids of constant curvature. He also remarked (as EULER had done) that a closed convex surface cannot be bent.

JACOBI also grasped immediately the importance of GAUSS' work in geometry and explained it in his lectures. C. G. J. JACOBI (1804-1851) was at that time at Königsberg. In a paper of 1836 he generalized GAUSS' theorem on geodesic triangles on a surface. (7) But JACOBI's contributions to differential geometry are much more important than this. He beautifully combined GAUSS' ideas with his own. Through his discoveries on Abelian integrals he was able to integrate the geodesic lines on an ellipsoid, which led to hyperelliptic integrals, a counterpart to MONGE's integration of the lines of curvature. (8) In the calculus of variations he established the existence of the conjugate points on the geodesics passing through a point on a surface.

(6b) F. MINDING, Bemerkung über die Abwicklung krummer Linien und Flächen. *Crelle* 6 (1830), p. 159.

(6c) F. MINDING, Wie sich entscheiden lässt ob zwei gegebene krumme Flächen auf einander abwickelbar sind oder nicht. *Crelle* 19 (1839), p. 370. In *Crelle*, from 1838-40, are several papers of MINDING on this subject, the remark on convex surfaces in 18 (1838), p. 365-368.

Through MINDING, his colleague K. E. SENFF (1810-1849) and their pupil, KARL PETERSON (1828-1881), Dorpat figures as a minor center in the history of differential geometry. SENFF found several results later discovered by SERRET. PETERSON, who later became teacher at Moscow, wrote a book " Ueber Curven und Flächen " (1868) with remarkable results, and in an unpublished " Kandidatenschrift " of 1853 found BONNET's theorem of 1866 on a surface being determined by first and second fundamental form and equations equivalent to those of MAINARDI. See P. STÄCKEL, *Bibl. mathematica* (3) 2 (1901), p. 122-132.

(7) C. G. J. JACOBI, Demonstratio et amplificatio nova theorematis Gaussiani de curvatura integra trianguli in data superficie e lineis brevissimis formati, *Crelle* 16 (1836) p. 344-358; see also note 10.

(8) C. G. J. JACOBI, *C. R.* 8 (1839), p. 284; *Crelle* 19 (1837), p. 309, Vorlesungen über Dynamik, p. 116.

On an ellipsoid these conjugate points form a curve with four cusps. (9)

In this work JACOBI touches on differential geometry in the large. Another example is his theorem that the spherical image of the principal normals to a closed continuously curved space curve divides the surface of a sphere into two equal parts, (10) a corollary of his generalization of GAUSS' theorem on geodesic triangles.

JACOBI regularly gave lectures on the theory of curves and surfaces, some of which still exist in manuscript. In KÖENIGSBERGER's book on JACOBI excerpts are given. They show a vivid interest in the production of the French school and GAUSS, and they present, with JACOBI's own work, much of this material in an original way.

An early pupil of GAUSS and JACOBI in Germany was F. JOACHIMSTHAL (1816-1861) to whose work we have to come back. A pupil of JACOBI at Königsberg was HEINRICH FERDINAND SCHERK (1798-1885), later teacher of mathematics at Kiel and Bremen, who found a minimal surface, differing from the two found by MEUSNIER more than fifty years earlier. (11)

One of the reasons for the small number of workers on differential geometry in those days was, of course, the great attraction exercised by the great algebraic geometers, STEINER, MÖBIUS, PLÜCKER, CHASLES, PONCELET, and others. STEINER and MÖBIUS occasionally ventured into infinitesimals, and the result was always interesting. For instance, STEINER found, in his original, thoroughly geometrical, way, the properties of the curvature centroid of closed plane curves (1838), and renewed in classical way the isoperimetrical problem (1841). A brief paper of 1839 deals with involutes of space curves; it is related to JACOBI's paper on space curves of that year. (12) MÖBIUS, in his *Barycentrische Calcül* (1827)

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(9) C. G. J. JACOBI, Posthumous paper : Über die Curve, welche alle von einem Punkte ausgehenden geodätischen Linien eines Rotationsellipsoides berührt. Ges. Werke VII, p. 72-80.

(10) C. G. J. JACOBI, Über einige merkwürdige Curventheoreme, *Astron. Nachr.* 20 (1842), p. 115-120. Ges. Werke VII, p. 34-39.

(11) H. F. SCHERK, Bemerkungen über die kleinste Fläche innerhalb gegebener Grenzen. *Crelle* 13, 1835, p. 185; earlier in a Latin paper in *Acta Soc. Jablonovianae* 4 (1830).

(12) J. STEINER, Einfache Beweise der isoperimetrischen Hauptsätze, *Crelle*

has three chapters on curves and surfaces, treated for the first time in an invariant symbolism. His work leads to DUPIN's results on the indicatrix, and he claims to be an independent discoverer of its properties; it was only after he had his results, he writes, that he discovered them in DUPIN's *Développements.* (13)

Accordingly, not only in Germany was there little immediate response to GAUSS' work, but also in France, where, moreover, the MONGE tradition was still very much alive. We may even feel a certain polemical note in a paper written by SOPHIE GERMAIN, (at that time already fifty-five), in *Crelle* 1831, recommending the "mean curvature" $\frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$ as the measure of curvature of a surface instead of GAUSS' $\frac{1}{R_1 R_2}$. The mean curvature is indeed of importance in elasticity, a field in which SOPHIE GERMAIN had won her laurels.

7. — *The French School of the Forties.*

The new generation in France considered GAUSS and JACOBI its leaders. GALOIS, in his last letter to a friend, expressed it in this way : " Tu prieras publiquement JACOBI ou GAUSS de donner leur avis non sur la vérité, mais sur l'importance des théorèmes." But there was also a second influence, working in France, namely the development of physics in general and mathematical physics in particular, — a direct result of the tremendous changes in the European industrial system.

During the Restoration this development, begun during the Empire, reached with FRESNEL and AMPÈRE a point where it veritably revolutionized scientific thought. Optics, electricity, elasticity, the theory of heat, and astronomy were affected. MALUS and CHLADLI were among the first venturing into these unknown regions; then followed POISSON, FOURIER, FRESNEL, ARAGO,

18 (1836), p. 281-296, Ges. Werke II, p. 75-91. Von dem Krümmungsschwerpuncte ebener Curven. *Crelle* 21 (1838), p. 33-63, 101-133, Ges. Werke II, p. 97-159. Ueber einige allgemeine Eigenschaften der Curven von doppelter Krümmung. *Monatsber. Akad. Berlin,* 1839, p. 76-80, Ges. Werke II, p. 163-165.

(13) A. F. MÖBIUS, Ges. Werke I, p. 1-388. Differential geometry in Ch. VI. VII, VIII. See Vorrede p. 8.

AMPÈRE, SADI CARNOT, CLAPEYRON, and many others. With NAVIER and SOPHIE GERMAIN, DE SAINT VENANT, and LAMÉ, the field of theoretical elasticity is opened, — an event which was to exert considerable influence on the application of analysis to geometry.

For our purpose the most important figure of this group is GABRIEL LAMÉ (1795-1870), a pupil of the Ecole Polytechnique, who was sent, in the company of CLAPEYRON, to Russia to assist in the improvement of methods of technical instruction. After the July revolution of 1830, he became professor at Paris. This engineering work in the development of new railroads exhibits clearly the close relationship which characterized the progress of natural science and the rapidly growing French capitalism of his time; compare, for instance, also the work of CRELLE on Prussian railroads. In asking for solutions of the equations of heat and elasticity in other than rectangular solids LAMÉ developed a theory of rectangular curvilinear coordinates, in which the line element of space is written

$$ds^2 = H^2\, d\rho^2 + H_1^2\, d\rho_1^2 + H_2^2\, d\rho_2^2$$

and where the quantity

$$\sqrt{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2 + \left(\frac{dF}{dz}\right)^2}$$

is what LAMÉ called the " paramètre différentiel du premier ordre " of the function $F\ (x, y, z)$. There are also " paramètres différentiels du second ordre "

$$\frac{d^2F}{dx^2} + \frac{d^2F}{dy^2} + \frac{d^2F}{dz^2}.$$

LAMÉ's six partial differential equations of the second order connecting the H form the basis of his investigations, which carry him deep into interesting analytical developments.

Here we find not only " coordonnées curvilignes " in space, but also invariants under rotations and translations. This invariance is indicated by the introduction of the symbols $\Delta_1\, F$ and $\Delta_2\, F$ for first and second differential parameters of $F\ (x, y, z)$. A single infinity of surfaces $F =$ const, for which $\Delta_1\, F = 0$, is called " isotherme," the function F a " paramètre thermométrique " of the family of surfaces.

LAMÉ introduced these notions in a series of publications, the first of which, in 1837, on isothermic surfaces, received immediate attention. It was followed in 1840 by a paper on curvilinear coordinates. (14) After many years of teaching he collected the fruits of his research in a series of textbooks, which are, even now, very readable. (15)

Another mathematician, who entered the field of differential geometry through his work in applied mathematics, was A. J. C. BARRÉ DE SAINT VENANT (1797-1886), well known as an early contributor to the theory of elasticity. He wrote, in 1846, a paper which supplied the impetus to the final completion of the elementary theory of space curves. (16)

It was under such teachers that the younger mathematicians developed. The leading figure in pure mathematics became JOSEPH LIOUVILLE (1809-1882).

LIOUVILLE's fame is the result, on the one hand, of his diversified mathematical investigations, and on the other, of his success as the founder and editor of the " Journal de mathématiques pures et appliquées." This Journal, first published in 1836, has published a great number of papers on geometry; its parallel in Germany was CRELLE's " Journal " of the same title. CRELLE and LIOUVILLE's Journals constituted the principal agencies for the progress of differential geometry in the middle part of the 19th Century, and as such their importance is not merely of the past.

Among the younger contributors to LIOUVILLE's Journal were J. A. SERRET (1819-1885), V. PUISEUX (1820-1883), O. BONNET (1819-1892), and J. BERTRAND (1822-1900), all of whom became professors at Paris, together with F. FRENET (1816-1868), professor at Lyons, and the Belgian E. CATALAN (1819-1894), who studied at the Ecole Polytechnique and became professor at Liége. Their work is based not only on that of LAMÉ and MONGE but also of GAUSS and JACOBI. We shall try to summarize, without going

(14) G. LAMÉ, Mémoire sur les surfaces isothermes, *Journ. de math.* (10)2 (1837), p. 149; Mémoire sur les coordonnées curvilignes, *ib.* 5 (1840), p. 313-347.

(15) G. LAMÉ, Leçons sur la théorie mathématique de l'électricité (1852); Leçons sur les fonctions inverses des transcendantes et les surfaces isothermes (1857); Leçons sur les coordonnées curvilignes et leur diverses applications (1859); Leçons sur la théorie analytique de la chaleur (1861).

(16) SAINT-VENANT, Mémoire sur les lignes courbes non planes. *Journ. Ec. Polyt.* cah. 30 (1846), p. 1-76.

too far into an estimation of individual merits, the results of this school, which flourished from 1840 to 1850.

1) *Work on Gauss' theorems.* In many papers and by various methods the theorem of GAUSS on the invariance of curvature and the sum of the angles of a geodesic triangle is proved again (17). It is in this work that modern nomenclature evolves more and more, as, R_1 and R_2, $ds^2 = Edu^2 + 2Fdudv + Gdv^2$, etc. BONNET names the " courbure géodésique," (18) and derives the " formula of GAUSS-BONNET." LIOUVILLE introduces as a counterpart to GAUSS' ds^2 in geodesic polar coordinates the $ds^2 = \lambda\ (du^2 + dv^2)$, already studied by GAUSS in his paper on map projections, and found in LAMÉ's investigations on heat (isothermal element). This leads LIOUVILLE to conformal representation and to " LIOUVILLE's surfaces," for which the geodesics can be found by quadratures. JACOBI's determination of the geodesics on an ellipsoid is a special case. (19) LIOUVILLE also determined all rotation surfaces of constant curvature, completing MINDING's results.

2) *Special problems in the Monge tradition.* MONGE's tradition remained entirely alive. As an example of work accomplished under its influence we mention several studies on surfaces with lines of curvature, plane or spherical. JOACHIMSTHAL, the German geometer, first at Halle, later at Breslau, shared in these investigations, (20) and found, among other results, the " theorem of JOACHIMSTHAL " on plane lines of curvature. In this connection we may also mention ABEL TRANSON (1805-1876), who continued the work of AMPÈRE and CARNOT and who belongs to the early students of affine differential geometry.

3) *The greater prominence of the notion of invariance.* The frequent use of curvilinear coordinates emphasizes GAUSS'

(17) E.g. J. LIOUVILLE, Sur un theorème de M. GAUSS concernant le produit de deux rayons de courbure principaux en chaque point d'une surface. *Journ. de Math.* 12 (1847), p. 291-304.

(18) O. BONNET, Mémoire sur la théorie générale des surfaces. *Journ. Ec. Polyt.* 19, cah. 32 (1848), p. 1-146.

(19) J. LIOUVILLE, Sur quelques cas particuliers où les équations du mouvement d'un point matériel peuvent s'intégrer, *Journ. de Math.* 11 (1846), p. 315. De la ligne géodésique sur un ellipsoide quelconque, *Journ. de Math.* 9 (1844), p. 401, also appendix of the fifth ed. of MONGE's " Applications."

(20) F. JOACHIMSTHAL, Demonstrationes theorematum ad superficies curvas spectantium, *Crelle* 30 (1846), p. 347-350.

standpoint on the invariance of quantities under coordinate transformations. LAMÉ's differential parameters $\Delta_1 F$ and $\Delta_2 F$ are a first effort to create a special notation for differential invariants on the surface. This development is contemporary with that in algebraic invariants, a field in which CAYLEY and ARONHOLD had already started their investigations. These geometers, however, did not arrive at the relationship between the GAUSS coordinates on a surface and the LAMÉ coordinates in space. This could be done only after RIEMANN's fundamental work on the nature of space.

4) *Development of the general theory of space curves.* The theory of space curves lacked both elegance and easy access. DE SAINT VENANT, who wrote an extensive study on space curves in 1846, (16) in which he collected the available material, added several new theorems, introduced the word " binormale," and gave a good historical review, concluding his paper with several pages of formulas. This approach could be abandoned when F. FRENET, in his Toulouse dissertation of 1847, found the key to an easy control of the computations in this field. (21) His " FRENET formulas " are the result of a belief that we should not only find the derivatives with respect to the arc of the direction cosines of the tangent, but also those of the principal normal (rectifying plane) and the binormal (osculating plane). A few years later J. A. SERRET, (22) ignorant of FRENET's dissertation, arrived at the same formulas; this led FRENET to a renewed publication of his results in the same periodical, *Liouville*'s Journal. Their importance was not generally recognized very soon. The textbook of F. JOACHIMSTHAL on differential geometry, which was the result of lectures at Breslau during the winter term of 1856-57, does not give them, (23) not even the third edition of 1890, edited by L. NATANI. Neither do they appear in a book on space

(21) F. FRENET, Sur les courbes à double courbure. Thèse Toulouse, 1847. Abstract in *Journ. de Math.* 17 (1852), p. 437.

(22) J. A. SERRET, Sur quelques formules relatives à la théorie des courbes à double courbure. *Journ. de Math.* 16 (1851), p. 193.

(23) F. JOACHIMSTHAL, Anwendung der Differential- und Integralrechnung auf die allgemeine Theorie der Flächen und der Linien doppelter Krümmung. Leipzig 1872. This excellent introduction, one of the first to appear in Germany, gives a symposium of the results of the French school and those of GAUSS and JACOBI.

curves, written in 1860 by another SERRET (PAUL) (1827-1898), a professor at the Université Catholique at Paris, who gave an interesting exposition of the whole theory as it was known in 1860, with special applications to spherical curves. (24)

5) *Solution of special problems on space curves.* PUISEUX proved that the ordinary helix is the only curve for which curvature and torsion are constant. (25) Then BERTRAND showed that the helix on a general cylinder is the only curve for which the ratio of curvature and torsion is a constant. SERRET found the equation of the curves of constant curvature and that of curves of constant torsion, and BERTRAND discovered the " BERTRAND curves," curves for which a linear relation exists between curvature and torsion. After SERRET had published his formulas, it was recognized that these results could be obtained much more easily from the FRENET formulas. PAUL SERRET's book of 1860 has already been mentioned.

6) *Original results.* It would be entirely wrong to think of these mathematicians only as pupils of great masters. As an example of a new and surprising result we mention LIOUVILLE's theorem, that conformal transformations in space are only inversions, similarity and congruency transformations. LIOUVILLE found this theorem in an investigation of the conformal representation of surfaces; he published it, together with many other results of his own and his associates, in the fifth edition of MONGE's *Applications*. LIOUVILLE also added a reprint of GAUSS' paper and as a result we get from this fifth edition a fair idea of the progress of differential geometry to the fifties of the nineteenth century. (26)

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(24) PAUL SERRET, Théorie nouvelle géométrique et mécanique des lignes à double courbure, 1860; also Thèse Paris, 1859, without some notes. The book also contains the theory of curves on surfaces, and has several new results, as e.g. the theorem that asymptotic lines of a ruled surface are cut by the generators in 4 points of constant anharmonic ratio (p. 169).

(25) V. PUISEUX, Problème de géométrie. *Journ. de Math.* 7 (1842), p. 65.

(26) A textbook showing the regular class programs of those days in France is C. F. A. LEROY, Analyse appliquée à la Géométrie des trois dimensions. Paris, Mallet Bachelier, 4th ed. 1854, 408 p., of which almost one half is devoted to differential geometry. It is almost exclusively based on the texts of MONGE and DUPIN, influenced by the presentation of these texts by CAUCHY and POISSON (*Journ. Ec. Pol.* XXI). The word " torsion " is used here (p. 292, 299). There is no influence of GAUSS.

8. — Riemann

Outside of France there was more interest than initiative. In England the interest in differential geometry could only be awakened after the fossilized epigones of Newton had been replaced by younger men, acquainted with Leibniz's methods. This renaissance was initiated by Charles Babbage and his friends in 1811. But scarcely any differential geometry was investigated till the publication of the *Cambridge Mathematical Journal* (1837), which soon became *The Cambridge and Dublin Mathematical Journal* (1846), in which geometry and analysis were cultivated in the continental style. Differential geometry, however, appeared mainly as an aspect of the theory of conic sections and quadrics. Among the authors in this journal we mention D. F. Gregory, W. Walton, A. Cayley, and W. Thomson. Gregory (1813-1844), who died young, left a well written *Treatise on the application of analysis to solid geometry*, edited by Walton (1845). It shows the French influence even in its terminology. The importance of Gregory is shown by the fact that his friends published his collected works. The centers of this new life were at Cambridge and Dublin, and at Dublin we find the leading figure of W. R. Hamilton (1805-1865), who published between 1828 and 1837 his four papers on systems of rays, which carry on the work of Monge and Malus and lead to his fundamental theorems on partial differential equations. (27) He also presented curve and surface theory in his quaternion lectures, published in 1853 and 1866. (28) They contain a complete theory of curves and surfaces, with many new results. He studies, for instance, space curves up to elements of the fifth order with respect to the arc-length. To his circle belong John Thomas Graves (1806-1870), a professor of law at London, who studied mathematics at Dublin, and the twin brothers Michael (1817-1882) and William Roberts (1817-1883), professors of mathematics at Dublin, who contributed to differential geometry. In the theory of confocal curves on an

(27) W. R. Hamilton, On systems of rays, *Trans. Roy. Irish Academy* 1828; Supplements I, II, ib. 1833; III, ib. 1837.

(28) W. R. Hamilton, Lectures on Quaternions, 1853; Elements of Quaternions 1866.

ellipsoid a theorem bears GRAVES' name. From this Dublin center come also the famous textbooks of the theologian-mathematician GEORGE SALMON (1819-1904), who deals to a considerable extent with the application of analysis to geometry, as in *Higher plane curves* (1852) and *Analytic geometry of three dimensions* (1862). England, though responsible for little creative work in our field, had, for a time, the better of the textbooks. In the forties we find in Germany only the work of F. JOACHIMSTHAL, an investigator who understood both the French and the German masters. In Italy differential geometry found excellent teachers in BORDONI and CHELINI, but it was their pupils who were to contribute first class results. Even in France little work of outstanding value is produced in the fifties. (29) A new impulse had to come.

This new impulse comes again from Germany, and again from Göttingen. In 1854, BERNARD RIEMANN (1826-1866) addresses an audience in the little town as part of the requirements for a " Dozentur." He talks " on the hypotheses which serve as foundation to geometry." GAUSS was in his audience. The paper was not printed till 1867, (30) but its publication set in motion an influence which has lasted till the present day.

The actual mathematical facts in RIEMANN's address can be stated in a few words. He develops the conception of a general n-dimensional manifold in the sense of the analysis situs and then introduces into it a quadratic linear element ds^2. He describes how the curvature of such a manifold can be measured, and specializes manifolds of constant curvature. In this way both Euclidean geometry and the non-Euclidean geometry of BOLYAY are obtained as special cases of a general " Riemannian " geometry. As these two cases are represented by a zero and negative constant curvature, the question arises as to a manifold of positive constant curvature, the " Riemannian" non-Euclidean spaces. The results of GAUSS on intrinsic properties of surfaces covered RIEMANN's special case $n=2$.

But the meaning of RIEMANN's work lies far deeper than the

(29) O. BONNET is an outstanding exception.

(30) Über die Hypothesen, welche der Geometrie zugrunde liegen, *Gött. Abh.* 13 (1867), Werke No. XIII, new edition by H. WEYL, Berlin, SPRINGER, 2 Aufl., 1921.

purely mathematical structure which resulted from it. We best see it perhaps when we compare MONGE and RIEMANN. MONGE studies surfaces in ordinary space. This space is for him a given entity, about which no discussion can exist. It is given and the task of the geometer is the same as that of the carpenter: he takes bodies in that space and works with them. There is a perfect separation of surface and space, because there is such a separation between rigid body and space. MONGE in this respect is the representative of the metaphysical materialism of the eighteenth century with its sharply defined conceptions.

RIEMANN does not only ask how rigid bodies behave in space, but also how space is affected by rigid bodies. Space is not the undiscussed given entity of MONGE, but rather an entity subject to the same scientific investigation as are the bodies themselves. Starting with RIEMANN, dialectical materialism begins to replace metaphysical materialism in the foundations of geometry. Space works on bodies, bodies influence space. There may be in the first " Anschauung " an undefined conception of space, but experience alone projects into it the properties which make it the space of our geometry and of our physics, the projective relations, the metrical relations and even the relations of analysis situs. From the abstract space of MONGE which relates bodies at finite distance from each other, we pass to the " field " theory of space as RIEMANN gives it. FARADAY and MAXWELL took the same step for the field of electricity (as WEYL remarks) (31). Also in other branches of science similar steps were taken at the same time, opening the passage from the metaphysical classification of eighteenth-century science to modern conceptions. The same dialectical methods characterize LYELL's work in geology, DARWIN's in biology, MARX' in sociology. From RIEMANN to EINSTEIN is but one step—a step however that had to wait sixty years. RIEMANN actually suggested, at the end of his paper, the possibility of determining the metric of space by the physical masses. This is the more striking as an example of almost superhuman divination, as RIEMANN's actual research in electricity and gravitation followed the classical lines of his day.

RIEMANN wrote but few formulas in his address. He had a

(31) See note (3), " Vorwort des Herausgebers."

chance to work out some of his ideas mathematically in a paper of 1861 on the distribution of electricity on cylinders. (32) Here he had to study the question of bringing the partial differential equation

$$\frac{\delta}{\delta x_1}\left(a_{11}\frac{\delta u}{\delta x_1}+a_{12}\frac{\delta u}{\delta x_2}+a_{13}\frac{\delta u}{\delta x_3}\right)+$$
$$\frac{\delta}{\delta x_2}\left(a_{21}\frac{\delta u}{\delta x_1}+a_{22}\frac{\delta u}{\delta x_2}+a_{23}\frac{\delta u}{\delta x_3}\right)+$$
$$\frac{\delta}{\delta x_3}\left(a_{31}\frac{\delta u}{\delta x_1}+a_{32}\frac{\delta u}{\delta x_2}+a_{33}\frac{\delta u}{\delta x_3}\right)=h\frac{\delta u}{\delta t}$$

a_{ik} functions of the x, into the simplest form. For a general number n of x_i this problem is equivalent to that of transforming the quadratic differential form $\Sigma\, a_{ii'}\, dx_i\, dx_{i'}$. RIEMANN finds the necessary and sufficient conditions that this form can be reduced to a sum of n squares in the form of the vanishing of the four index symbols.

$$(ii',\, i''\, i''') = 0.$$

Here, therefore, we have the principal concomitant of Riemannian geometry.

It is, however, underestimating RIEMANN's influence on differential geometry by referring only to his work on the space problem. His work on function theory opened new ways of attacking geometrical problems, as is shown, for instance, by the later investigations of SCHWARZ on minimal surfaces. RIEMANN's introduction of the connectivity of a manifold plays an important role in later differential geometry in the large.

Another study on n-dimensional geometry, from an entirely different point of view, appeared at the same time, or better, reappeared. That was GRASSMANN's *Ausdehnungslehre*. HERMANN GRASSMANN (1809-1877), who was a high school teacher (Gymnasiallehrer) at Stettin, had written a first edition of this work in 1844, which remained almost entirely unknown. He, therefore, rewrote it entirely, and published it again in 1862. (33)

(32) Commentatio mathematica qua respondere tentatur quaestioni ab Ill. Academia Parisiensi propositae, 1861, Werke, p. 391-423, only published posthumously.

(33) H. GRASSMANN, Die lineale Ausdehnungslehre. Ein neuer Zweig der Mathematik, 1844. Ges. Schriften I (Leipzig, Teubner), p. 1-139. Die Ausdehnungslehre 1862. Ges. Schriften II, p. 1-511.

Here we find an admirable treatment of the elementary Euclidean (and affine) n-dimensional geometry in the form of a (far from elementary) formal calculus of points and vectors, as "Ausdehnungsgrössen," a consistent direct calculus invariant under the affine group and partly under the rotational group. Not only vectors are introduced, but also what we now call tensors. This work has not only bearing on differential geometry in so far as it inspired other geometers to extend their investigations to more dimensions, but also through the direct application GRASSMANN himself made of his apparatus to the problem of PFAFF, through which this problem first took geometrical form.

RIEMANN certainly is the originator of more-dimensional differential geometry, GRASSMANN however of the symbolical methods introduced for the study of this geometry. Here GRASSMANN's work is supplemented by that of HAMILTON on quaternions. HAMILTON also introduces a symbolical method, of which the vector and scalar product, and the nablaoperator, have been utilized in the modern treatment of differential geometry. The invariant character of differential geometry, clearly expressed by GAUSS and the LIOUVILLE school, has its formal expression here, as is immediately seen by the easy way in which LAMÉ's differential parameters fit into the HAMILTON scheme. HAMILTON, however, has no "Ausdehnungslehre."

RIEMANN and GRASSMANN together form a remarkable example of the ways of development of human knowledge, where one mind, essentially dialectical in nature, hews down barriers and discovers new relationships, and the other, operating more formally, builds up new symbolisms to control the field again through rigid, even frozen, methods. This reveals again the greatness of LEIBNIZ, who, in this respect at any rate, combined the merits of RIEMANN and GRASSMANN.

9. — *The beginning of modern times.*

From now on the development of differential geometry does not follow one main line. In the second part of the nineteenth century different centers and different schools arise, in which the influence of their leaders extends far beyond their immediate location. Three sources of creative study draw immediate attention, though many different nations participate. These three sources are France, Germany and Italy. Their paramount positions did not arise accidentally. France with its long established traditions was the oldest center and continued uninterruptedly its economic development under Bonapartism, and later under the Third Republic. In Germany and Italy we see, at the same time, the birth of the national state, completed in 1870, and, as a result, an enthusiastic expansion of the capitalist system. The creative influence of this development on mathematical thought is clearly seen in Italy.

Here the national revival was known as the Risorgimento. After a series of wars against Austria and its own despots, the Italian people established the kingdom of Italy.

For a long time there had been much interest in differential geometry, mainly in the North, at the University of Pavia, where it was especially fostered . Here A. M. BORDONI (1789-1860) taught for many years, and published, beginning in 1821, papers on the applications of analysis to geometry, papers which show the influence of GAUSS and the LIOUVILLE school. But his important qualities lie mainly in his power to create interest. As colleague he had G. MAINARDI (1800-1879), as pupils or colleagues, D. CODAZZI (1824-1875), F. BRIOSCHI (1824-1897), L. CREMONA (1830-1903), F. CASORATI (1835-1890), and above all, E. BELTRAMI (1835-1900). Another early geometer of influence was the priest D. CHELINI (1802-1878), who was professor of mathematics and physics at different places, and who stimulated interest through his publications (commencing 1845) and his teaching.

Productive activity rises to a higher level in the fifties, when MAINARDI publishes a paper (34) in which he inquires as to what relations must exist between the six functions E, F, G, D, D', D'' introduced by GAUSS into the theory of surfaces. It is clear that there must be such relations, because the coordinate representation of a surface shows that three functions determine the surface. GAUSS found one relation, and the task is to find others. This leads MAINARDI to four equations, in which we now

(34) G. MAINARDI, Su la teoria generale delle superficie, *Giorn. Istit. Lombardo* 9 (1856), p. 385-398.

recognize the two "Mainardi-Codazzi" equations. At the same time the younger generation begins to publish, BRIOSCHI in 1852, CREMONA in 1855, and CODAZZI in 1858.

All these younger mathematicians were not only scholars, but also organizers and prominent in political activities. They were, accordingly, in a position to influence not only their pupils, but also the instruction of the whole country and build up, in a relatively short time, remarkable scientific enthusiasm. In geometry, Italy became one of the leading countries.

In the sixties France, Germany and Italy begin to participate equally in the development of differential geometry. A problem set by the Paris Academy acted as a stimulus : to study the applicability of surfaces upon each other (1860). This problem, a consequence of GAUSS' theory, had so far commanded the attention of MINDING and BONNET in France, who had framed the problem as that of recognizing when two given surfaces are applicable, that is, when they have the same ds^2. Answers to the Paris question came from BOUR and BONNET in France and from CODAZZI in Italy; WEINGARTEN in Germany wrote a related paper. Here we find the problem as that of finding all surfaces which are applicable to a given surface. EDOUARD BOUR (1832-1866), after whom this problem is called, and who got the first prize, was a young mathematician at Paris, who became professor at the Ecole Polytechnique ; his paper, in the words of LIOUVILLE, "peut être pris comme un beau mémoire de LAGRANGE." (35) BONNET, whose work covers the whole period in which differential geometry took its modern shape, wrote more than one important paper during this period; in 1867 he reached the fundamental result that a surface is perfectly determined, but for its position in space, by its first and second fundamental form, if these forms are related by CODAZZI's equations. (36) CODAZZI's contribution to surface theory (37) was followed by other papers in which

(35) E BOUR, Théorie de la déformation des surfaces. *Journ. Ec. Polyt.*, 22, cah. 39 (1862), p. 1-148.

(36) O. BONNET, Mémoire sur la théorie des surfaces applicables sur une surface donnée, *Journ. Ec. Polyt.*, 24 (cah. 41), 1865, p. 209-230; 25 (cah. 42, 1867), p. 1. See on BONNET the necrology by P. APPELL. *Comptes rendus* 117 (1893) p. 1014-1024.

(37) D. CODAZZI, Mémoire relatif à l'application des surfaces les unes sur les autres. *Mém. prés. par div. sav.* 27 (1882), Nr. 6; it was written in 1859.

he went deeper into the nature of certain equations used by him in his Paris answer, and he showed how they led to the three " Mainardi-Codazzi equations." He did not know that MAINARDI had reached similar results before. The problem of the applicability of surfaces leads naturally to these equations. BOUR gave them, but only for geodesic polar coordinates.

JULIUS WEINGARTEN (1836-1910), from 1873 to 1903 professor at the Technische Hochschule of Berlin-Charlottenburg, developed in his early papers the theory of the so-called W-surfaces, for which a relation exists between the principal radii of curvature. As the problem of finding all surfaces isometrical to a given rotation surface can be reduced to that of finding all W-surfaces of the same class, WEINGARTEN gave an example of a complete set of isometrical surfaces that can be found only by elimination and quadrature. Before that time only the developable surfaces were known. All the later work of WEINGARTEN is related to this problem of the applicability of surfaces. (38)

The important papers of this period are all of the type characterized by the names LAMÉ and GAUSS, and consciously or unconsciously reflecting the ideas of RIEMANN. The BOUR-MINDING type of problem set by GAUSS is indeed an investigation into the nature of two-dimensional RIEMANN manifolds; the same is true of the BONNET problem on the congruency of surfaces, and the investigations of THEODORE MOUTARD (1827-1901) on infinitesimal isometry. Then we have the fundamental papers of BELTRAMI, who, between the years 1864 and 1868, starting from LAMÉ, led directly into the problems left open by RIEMANN. Finally we have papers immediately due to RIEMANN, as HELMHOLTZ' inquiry into the foundations of geometry, and CHRISTOFFEL and LIPSCHITZ' study of the transformation of quadratic forms.

Several German mathematicians whose main line of work lies

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(38) D. CODAZZI, Sulle coordinate curvilinee d'una superficie e dello spazio. I, II, III, *Annali di Matem.* (2) 1 (1867-68), p. 310, 2 (1868-69), p. 101-119, 269-287, " Codazzi-equations ", p. 273-274. See on this subject R. v. LILIENTHAL, *Encykl. d. math. Wiss.* III, 3, p. 159.

J. WEINGARTEN, Über eine Klasse aufeinander abwickelbarer Flächen, *Crelle* 59 (1861), p. 382.

See also J. WEINGARTEN, Ueber die Oberflächen, für welche einer der beiden Hauptkrümmungshalbmesser eine Function des andern ist, *Crelle* 62, p. 164; *Crelle* 59, p. 382.

outside this field contribute in these years important papers to differential geometry. KUMMER writes on line congruences, WEIERSTRASS on minimal surfaces, and HELMHOLTZ on the axiomatics of geometry.

EUGENIO BELTRAMI is the most brilliant representative of the group of Italian geometers who were trained during the Risorgimento. He was, from 1862 on, professor at Bologna, Pisa, Bologna, Rome (1873-1877), Pavia, and Rome (1891-1900). One of his first papers was a translation of a work of GAUSS. During his time at Pisa he often met RIEMANN, and his work shows the deep influence of both GAUSS and RIEMANN. In an astonishing tempo he published his fundamental research on differential geometry; in 1864, he produced his extension of LAMÉ's differential parameters to curvilinear coordinates with references to the applicability problem; in 1865, several beautiful theorems on the bending of surfaces; in 1867, the generalization of complex functions to surfaces; and in 1868, his theorems on non-Euclidean spaces, in which he happily combined RIEMANN's ideas with the older notions of GAUSS and LOBATCHEVSKI. Here he showed that there is a representation of non-Euclidean geometry on the pseudosphere, so that there is no possibility of a contradiction in non-Euclidean geometry, and he proved that in manifolds of constant RIEMANN curvature the geodesics can be expressed by linear equations. (39) SCHLÄFLI (1814-1895), from 1852-1891 professor at Bern, then proved the inverse theorem. (40) This brought RIEMANN's ideas definitively out of the realm of speculation into the main body of mathematical investigation, and it resulted in the combination of the ideas of LAMÉ, GAUSS and RIEMANN into one solid theory.

The theory of quadratic forms of *n* variables was taken up independently by CHRISTOFFEL and LIPSCHITZ in papers published in 1869 in the same periodical, CRELLE's " Journal ". ERWIN

(39) Some of these papers are E. BELTRAMI : Ricerche di analisi applicata alla geometria, *Giorn. di mat.* 2 (1864), 3 (1865); Sulla teoria generale delle superficie, *Atti Ist. Veneto* (215 (1860); Delle variabili complesse sopra una superficie qualunque, *Ann. di Mat.* (2) 1 (1867); Teoria fondamentale degli spazi di curvatura costante, *Ann. di Mat.* (2) 2 (1868-69); Saggio di interpretazione della geometria non-euclidea, *Giorn. di Mat.* 4 (1868). All these papers in his " Opere," I, II.

(40) L. SCHLÄFLI, Nota alla memoria del sig. BELTRAMI. *Ann. di mat.* (2) 5 (1871-73), p. 178-193.

BRUNO CHRISTOFFEL (1829-1901), at that time at Zürich, became, after the Franco-Prussian war, professor at the new German university of Strassburg, where he taught with REYE. He also published some papers on related subjects, as on geodesic triangles on a surface and the determination of a surface by its mean curvature. RUDOLF LIPSCHITZ, (1832-1903) since 1864 professor at Bonn, who is also well remembered in analysis, continued his investigations on Riemannian manifolds during the next years and built up a theory of these manifolds; since RICCI's work these papers are somewhat antiquated. Both geometers established theorems on the equivalence of quadratic forms and derived the so-called Riemann-Christoffel tensor. To this purpose CHRISTOFFEL constructed his " CHRISTOFFEL symbols."

HERMANN HELMHOLTZ' (1821-1894) paper is *Ueber die Tatsachen, die der Geometrie zum Grunde liegen* (1868). (41) The name already shows the influence of RIEMANN's address of 1854, that was published a short time before HELMHOLTZ composed his paper. He emphasizes the empirical aspects of our space conception, in the way RIEMANN had indicated. HELMHOLTZ, a physiologist, had been led to the same idea by his own investigations. Experience, he says, gives us several measurable quantities, continuous and of more dimensions, such as space, the system of colors, and the field of vision. Each more-dimensional continuum is characterized by its own special properties, and our task is to analyze these properties. HELMHOLTZ now gives four postulates by which he can get RIEMANN's geometry with the quadratic line element; they are : 1) the existence of n dimensions and of continuity; 2) the existence of moving rigid bodies; 3) the free movability of a rigid body; 4) no dependence of the form of a rigid body on rotations (the monodromy). He then proves as a purely mathematical theorem that RIEMANN's geometry is the only case in which these conditions are satisfied.

This paper by HELMHOLTZ (later corrected by LIE) was of great importance for the final adaptation of non-Euclidean geo-

(41) *Gött. Nachrichten* 1868, Nr. 9; *Wiss. Abhandlungen* II, p. 618-639. It was in 1866 preceded by an address " Ueber die tatsächlichen Grundlagen der Geometrie," *Wiss. Abh.* II, p. 610-617. Later HELMHOLTZ defended his thesis against the philosopher LAND: " Ueber den Ursprung und Sinn der geometrischen Sätze." *Mind* 10 (1878), p. 212-224, *Wiss. Abh.*, II, p. 640-660.

metry. But, like RIEMANN's paper it was more than that. It was one of the outstanding contributions to an understanding of the nature of space, which was brought from the field of speculation into that of experimental physics. As such it is one of the outstanding contributions to a materialistic conception of space.

DARBOUX' work will be discussed in the next paragraph. To this period belong also investigations of ALFRED ENNEPER (1830-1885), from 1859 Dozent, after 1870 professor at Göttingen. He gave his name to a minimal surface and to a theorem on the torsion of the asymptotic lines. The first papers by H. A. SCHWARZ (1843-1921) on minimal surfaces appeared after 1865; his final solution of the isoperimetrical problem for the sphere dates from 1884.

The ideas of LAMÉ were taken up in a different way by French geometers who now began to study triply-orthogonal systems. LAMÉ, following DUPIN, had used only such systems as enabled him to integrate his differential equations of physics. The question of the construction of such systems was still unsolved. BONNET, in a paper of 1862, opened it anew; then follow A. RIBAUCOUR (1845-1923), E. COMBESCURE (1824-1889), and DARBOUX (thesis), (42) who showed respectively the relation of these systems to circle congruences, the way in which such systems can be obtained by a transformation of a surface into other surfaces with parallel tangent planes, and their dependence on a differential equation of the third order.

To the work done in the older line we mention papers written by LOUIS AOUST, a priest, " chanoîne de Montpellier " (1814-1885), who was professor of mathematics at Marseilles since 1854 and contributed largely, in papers and textbooks, especially between 1860 and 1880, to the theory of curves and surfaces; in these his " courbure inclinée " plays a large, too large, a role.

REINHOLD HOPPE (1816-1900), in 1859 Dozent, and after 1870 professor at Berlin, also published many papers, most of them antiquated now; several however are of importance as first ventures into the differential geometry of *n* dimensions. His work on intrinsic coordinates of curves deserves mention.

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(42) See the discussion by E. SALKOWSKI. *Encykl. d. math. Wiss.* III, 3, p. 541. See for the work done in France J. BERTRAND. Rapport sur les progrès les plus récents de l'analyse mathématique. Paris, 1867.

Several textbooks appear at this time. We already have mentioned PAUL SERRET's book. In 1868 appeared PETERSON's monograph *Über Kurven und Flächen*, in which the Moscow mathematician published several interesting results on applicability of surfaces (see footnote 6c). In O. BÖKLEN's *Analytische Geometrie des Raumes* of 1861 is a discussion of infinitesimal geometry. WILHELM SCHELL's (1826-1904) *Allgemeine Theorie der Kurven doppelter Krümmung* of 1859, was written under the influence of JACOBI, whose lectures on differential geometry he had heard in the winter of 1849-50. Some of these books were more monographs on certain subjects than complete textbooks. A good complete textbook on differential geometry had not yet been written.

10 — *Differential geometry from* 1870 *to* 1900.

From 1870 to the world war there were no important economic disturbances in Western Europe. Life at the expanding and flourishing universities went on very regularly, and mathematicians could devote their full attention to professional problems. A result of this long period of quiet development was a great progress in differential geometry, mainly along paths blazed by the previous generations. At the same time geometry became more and more an abstract science, which did no longer easily reveal its origin in practical problems. Most geometers of this period adopted the attitude of JACOBI that science exists for the glory of the human mind and that its primary importance lies outside the field of the applications. To this attitude we owe the development of differential geometry along lines of abstract beauty often comparable to that of the most unpractical of all sciences, the theory of numbers. France, Germany and Italy remained the chief centers, where at the universities differential geometry was taught and professed as a regular part of the schedule. Other countries participated in the development, sometimes in a very important way, as Norway with S. LIE, Sweden with A. BÄCKLUND, and Russia with P. L. TSCHEBYCHEF. By the end of the century the United States begins to exert some influence. It is impossible to give a discussion of all the different subjects taken into consideration, of the theories developed, and of the papers published during this period. This would, moreover, repeat in an unsatisfactory way what volume 3[3] of the *Encyklopädie der mathematischen Wissenschaften* in a series of monographs has done very effectively. We shall only point out some very general trends.

At the opening of the period we have the early work of DARBOUX, KLEIN and LIE, influenced by C. JORDAN and the so-called Erlangen Program of KLEIN (1872). (43) Here the consequence is drawn from the geometrical investigations of the nineteenth century, and the group concept is found to be the underlying idea. A geometry is defined as the theory of invariants belonging to a given continuous group of transformations. Also in differential geometry this idea of transformation comes more and more into prominence. It finds its nucleus in EULER, and later in the work of MONGE, CARNOT and AMPÈRE, it arises again in the work of GAUSS, LAMÉ and RIEMANN, and finally in that of BELTRAMI. The Erlangen Program became the program indeed of almost all further work on geometry in the nineteenth century, revealing itself not only in opening of new fields of investigations, but also in the introduction of formal invariant methods into the exposition of the material.

Nevertheless, the Erlangen Program did not exhaust the field of geometry as a whole, and so of differential geometry. There have always been tendencies leading outside of the vigorous scheme by which KLEIN so successfully tried to lead the development of geometry. RIEMANN's manifold conception was broader, and Riemannian geometry can only fit into a wider frame than that of the Erlangen Program.

The three great geometers whose work dominates the period after 1870 are GASTON DARBOUX, SOPHUS LIE and their younger contemporary, LUIGI BIANCHI. LIE's work consciously lies in the school of the Erlangen Program, for which he is partly responsible. BIANCHI and DARBOUX, though their work belongs to the type considered by KLEIN, did not find their field of work in problems especially suggested by it. The conscious cultivation of the differential geometry of different continuous groups belongs principally to the twentieth century.

G. DARBOUX's (1824-1917) career is analogous to that of many

(43) F. KLEIN, Vergleichende Betrachtungen über neuere geometrische Forschungen, Programm Erlangen 1872, *Math. Annalen*, 43 (1893), p. 63-100.

French geometers of the nineteenth century, his life being that of scholar and teacher at Paris. A remarkable fact is that for his education he was one of the first to prefer the Ecole Normale to the Ecole Polytechnique. In his later years, as secrétaire perpétuel of the Academy he exercised influence far beyond the limits of his special field of work. His work, even more than that of L. BIANCHI (1856-1928), professor of mathematics at Pisa and pupil of BETTI and DINI, follows the classical lines and shows the enormous variety of problems left unsolved by MONGE, LAMÉ and others. DARBOUX's early work (beginning 1866) deals with orthogonal sets of surfaces, each family of surfaces necessarily depending on a partial differential equation of the third order. Then he passes to a study of the deformation of surfaces, to the applicability of surfaces, and to surfaces of constant curvature. BIANCHI's productive career begins with the applicability of surfaces (1878) and in rapid succession follow papers on surfaces of constant curvature, orthogonal surfaces, Weingarten surfaces, minimal surfaces, and congruences. BIANCHI also turns to non-Euclidean geometry, taking up the ideas of BELTRAMI; the surfaces of curvature zero in such geometry draw his special attention. Following BÄCKLUND he invents transformations to pass from one set of surfaces of special character to another. In the work of DARBOUX and BIANCHI partial differential equations play an important role. As a new aspect of the theory of curves on surfaces, DARBOUX introduces the coordinates x, y, z, of a surface as solutions of an equation

$$\frac{\delta^2\theta}{\delta\alpha\delta\beta} + A\frac{\delta\theta}{\delta\alpha} + B\frac{\delta\theta}{\delta\beta} + C\theta = 0,$$ A, B, C functions of α,β,

α=const, β=const appear then as conjugate parameter lines. In DARBOUX's work another characteristic appears, the "trièdre mobile," by which kinematical considerations are introduced into the study of curves and surfaces. This field, opened by CODAZZI, was especially cultivated by A. MANNHEIM (1831-1906).

The principal fame of DARBOUX and BIANCHI lies in their beautiful textbooks, in which they combined their own results with those of their predecessors. BIANCHI's *Lezioni di geometria differenziale* (1893), a new edition of autographed lectures published in 1886, is a systematic treatise of the theory of curves and surfaces, with special attention paid to more-dimensional geometry.

It became known in wider circles through the German translation of M. LUKAT (1899). DARBOUX's *Leçons sur la théorie générale des surfaces* (four volumes, 1887, 1889, 1894, 1896) is much more than the title professes; it is an exposition of a vast number of theories related to the theory of surfaces, connected in a loose way, so as to give the author an opportunity to demonstrate the enormous breadth of his knowledge and the elegance of his methods.

For SOPHUS LIE (1842-1899) the transformation group is not an important means of investigation alone; it is rather the central part of geometry. His work is closely connected with that of KLEIN. He was a clergyman's son from a village in Norway, and he visited Paris in 1869-70 together with KLEIN, where he exchanged ideas with C. JORDAN and DARBOUX. JORDAN, at that time, had taken up GALOIS' result and published the *Traité des substitutions*. During their travel the fundamental importance of the group became clear to KLEIN and LIE. A first result was a joint paper on W-curves, plane curves invariant under a projective transformation of the plane. The Erlangen Program cast their ideas into permanent form. LIE then started out on his life work, the study and classification of continuous transformation groups, an inquiry which at the same time opened a new road to the study of partial differential equations. Regularly he reached results of importance to differential geometry. We already had occasion to compare his combination of geometrical intuition with analytical skill to that of MONGE. In his hands more-dimensional considerations grew into as natural a part of geometry as the classical methods. Of his special contributions we may mention the duality between line and sphere geometry leading to the duality between asymptotic curves and lines of curvature, the integration of the FRENET formulas if a relation between curvature, torsion and arc length is given, his work on minimal surfaces and contact transformations. He improved HELMHOLTZ' analysis of RIEMANN's problem of space. Of his older pupils we mention F. ENGEL and W. KILLING.

More-dimensional differential geometry found also investigators following the directives of RIEMANN, CHRISTOFFEL and BELTRAMI. Through a series of investigations undertaken by many authors a complete theory of curves and higher manifolds was developed. This study was considerably simplified by the invention of the

" calculo differenziale assoluto," a symbolism to express the invariants of Riemannian geometry. The inventor was GREGORIO RICCI-CURBASTRO (1853-1924), as BIANCHI pupil of BETTI and DINI at Pisa, since 1880 professor at Padua. RICCI published his discoveries in a series of papers beginning in 1886.

An entire new field is represented by the work of HENRI POINCARÉ (1854-1912). In a series of papers under the title : *Mémoire sur les courbes définies par une équation différentielle*, published in the *Journal de mathématiques* from 1881-1886, the great physicist, astronomer and analyst undertook the study of the properties of integral curves of ordinary differential equations, their singular points and their behavior in the large. (44) Again differential geometry was enlarged by influences from outside fields, as in the times of GAUSS and of LAMÉ. POINCARÉ was led to his theory by his investigation on problems similar to the three-body problem and related questions of dynamics, where geodesics appear as trajectories of moving particles. Here appear singularities like the " nœud," the " col," the " centre ", and the " foyer," and relations connecting them. Analysis situs becomes definitely connected with differential geometry. Differential geometry in the large, so far appearing only in isolated remarks of EULER, GAUSS, JACOBI, MINDING, turns into an important and difficult science of its own. The question of surfaces with closed lines of a certain kind arises.

The main tendencies of the years 1870-1900 can be summarized as follows :

a) consequent study of continuous transformation groups;

b) triply orthogonal systems;

c) surfaces of constant curvature;

d) applicability and deformation of surfaces;

e) renewed study of partial differential equations and their geometrical interpretation.

f) integral curves of ordinary differential equations;

g) more-dimensional geometry;

h) adaptation of invariant methods, as vector analysis or absolute differential calculus;

i) conception of comprehensive textbooks related to the general

(44) See the discussion by H. LIEBMANN, *Encykl. d. math. Wiss.*, III, 3, p. 503.

acceptance of differential geometry in the mathematical programs of the universities.

There are, of course, many aspects to this rich development that we have failed to emphasize or even to mention. The work of E. CESARO (1859-1906), professor at Naples, on natural geometry, resulting in a well-known textbook (1896) belongs to these. It may be characterized as one of the attempts to introduce invariant methods. Other mathematicians began to introduce vector methods into differential geometry, as G. PEANO (1887), or C. BURALI FORTI (1897); other attempts, like that of J. KNOBLAUCH (1888) may also be classified under this heading.

A regular output of textbooks showed how general differential geometry was taught. Almost all the " Cours d'analyses " and corresponding books in other languages devoted chapters to curves and surfaces. Since the seventies special books on the subject appeared regularly, from the books by the Abbé AOUST to those of L. RAFFY (1897), G. RICCI (1898) and W. DE TANNENBERG (1899). By the end of the century differential geometry had begun to break up into special branches, each of which had its specialists.

This leads us to the beginning of the twentieth century, when other tendencies begin to exert influence. We shall not deal with them in this present outline.

11. — *Sources.*

The material, necessary for a history of differential geometry, is scattered over many publications, and so far as we know has never been reviewed as an historical whole. Up to 1800 we have V. KOMMERELL's excellent paper on *Raumkurven und Flächen* in the fourth volume of M. CANTOR's *Vorlesungen über die Geschichte der Mathematik* (1908). After 1800 we have F. KLEIN's monumental book on *Entwicklung der Mathematik im 19. Jahrhundert* I (1926), but this deals with only a few aspects of differential geometry. These are the only larger expositions on the subject. Of importance also are A. HAAS, *Versuch einer Darstellung der Geschichte des Krümmungsmasses* (Diss. Tübingen, 1881), not always correct, and S. A. CHRISTENSEN, *Om den historiske udwikling af theorien for fladers og rumkurvers krumning* (Tidskrift f. Math., 1, 1883, p. 97-127). Then there are several monographs, as P. STÄCKEL, *Bemerkungen zur Geschichte der geodätischen Linien*," (Leipz. Berichte 45, 1893, p. 444-467), (45) and many biographies, of which the " Eloges " of ARAGO, BERTRAND and DARBOUX contain fascinating details, and are both valuable and charming.

12. — *Final remarks.*

From time to time we have been able to refer, in a few words, to the influence of external factors in the development of differential geometry. There is a general tendency here.

Nobody seems more free than the mathematician in the selection of his problems; yet even he in this respect has no complete liberty. He generally follows the directives of a certain school of thought, opened by a leader in the field, and is guided in his work by established traditions. The leader, however, is himself under certain influences in the selection of his material, and the direction of his labor is, as a rule, determined by external factors. In all cases his topics of research are in some way or another connected with the material conditions under which his generation lives, either directly by the necessity of the mathematical treatment of a technical problem, or indirectly by certain philosophical principles estimated as valuable in his time. Even when freedom of selection seems the very nature of his mathematical work, it is material circumstances which allow the required amount of abstraction from daily needs, as in the later part of the nineteenth century under the " science for science's sake " attitude prevalent at many universities, and reaction is bound to follow closely. We, therefore, find a subtle, but nevertheless definite, relationship between the general economic problems that humanity has had to solve and the investigations of the mathematician. It is, therefore, possible to indicate these connections even in the case of one of the most abstract fields of mathematics, the application of analysis to geometry. Though this field is too large to admit more than a very insufficient treatment in a few pages, we were nevertheless

(45) Also a master's thesis at the Mass. Inst. of Technology by J. L. LAWSON. The history of the theory of curves and surfaces as contained in papers presented before the French Academy of Science during the entire eighteenth century, 33 p., 1931.

able to give some indications of how the different schools of geometers have been influenced by the technical requirements of their age.

Differential geometry owes its results to such problems as are set by map projection, surveying, measurement of time (HUYGENS), fortification and other problems of warfare (MONGE); these are supplemented by problems taken from potential theory, elasticity, light and vision. Its very origin is due to the invention of the calculus, a result of a long series of involved technical problems in a period when geometry was still the main form of mathematics. The French revolution opened enormous sources of energy for its promotion, and influenced that renewed study of space and time connected with the names of KANT, GAUSS and LOBATCHEVSKY. The steady penetration of materialistic ideas into the study of separate branches of science in the course of the nineteenth century resulted in the realistic analysis of the space problem by RIEMANN and HELMHOLTZ, and removed the previously existing barriers to the study of *n*-dimensional geometry. For the organization and further cultivation of differential geometry the nationalistic revival of the sixties was of great importance, as is shown, for example, by the Risorgimento in Italy. And the unhampered development of science in the latter part of the last century, which resulted in a steady growth of our knowledge of curves and surfaces, also finds its source in the undisturbed material conditions of Europe during that time.

These are a few of the tendencies that have been responsible for the development of one of the most fascinating branches of modern mathematics. In the preceding exposition we had to make a selection, which was to a certain extent arbitrary, and which depended on our personal preferences. Nevertheless, we hope we have followed what most mathematicians will consider the main lines of development.

Massachusetts Institute of Technology *February* 1932.
Cambridge, Mass. D. J. STRUIK.